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ESO 3 Math DYO Initiative

PAN 3 Formative Assessment/Instruction Cycle in Math Research Study

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ESO 3 Math DYO Initiative
PAN 3 FORMATIVE ASSESSMENT/INSTRUCTION CYCLE IN MATH
RESEARCH STUDY
October, 2010

Table of Contents

I.	EXECUTIVE SUMMARY: WHAT DID WE LEARN?	1
II.	DESCRIPTION OF THE STUDY	4
	Introduction.....	4
	Research Design.....	5
III.	BACKGROUND ON MATH IN THE CITY	6
IV.	FINDINGS.....	8
	Core Practice: DYO Math Assessment.....	8
	Core Practice: Instructional Implications Work	13
	Core Practice: Instructional Planning	20
	Core Practice: Classroom Instruction	24
	Outcome: Student Demonstration of Mathematical Thinking.....	34
V.	REFERENCES	42
VI.	APPENDICES	43
	Appendix A: Data Collection Instruments.....	43
	Appendix B: Data Coding and Analysis Instruments	47
	Appendix C: Action Researcher Lesson Observations and Analyses	49

I. EXECUTIVE SUMMARY: WHAT DID WE LEARN?

This report presents the findings from the study conducted by the NYC DOE Empowerment Network 3 and the National Center for Restructuring Education, Schools, and Teaching (NCREST), Teachers College, Columbia University on the Practice Area Network 3 (PAN3) Formative Assessment/Instruction Cycle in Math. The study examined teacher participation in the PAN3 Formative Assessment/Instruction Cycle in Math and its impact on:

- a) the design, development, administration and analysis of DYO assessments in math.
- b) teacher understanding of formative data on students' mathematical thinking and their instructional implications.
- c) teacher use of formative data on students' mathematical thinking to plan math instruction.?
- d) teacher classroom practices in math instruction.
- e) the development of third grade students' mathematical thinking.

What follows is a summary of major findings from each of the four component core practices of the PAN 3 Formative Assessment/Instruction Cycle in Math, as well as student demonstration of mathematical thinking outcomes.

Core Practice: DYO Math Assessment

This core practice entails the design, administration, and analysis (e.g., scoring or sorting) of DYO math assessments aimed at eliciting student mathematical thinking. We learned that:

- of those teachers and action researchers who are part of the DYO assessment design team, most reported that participating in the assessment design impacted their classroom practices, specifically in connection to developing other assessments and to broadening their ideas about teaching mathematics.
- a few teachers expressed concern about the timing and content of the context/story problems of the DYO assessment, noting that when this part of the DYO does not correspond to what they are teaching at a particular time, their ability to effectively use the DYO data to make instructional decisions is limited.

Core Practice: Instructional Implications Work.

This core practice relates to the convening of teachers, administrators, and instructional coaches to review student DYO work and discuss and articulate instructional implications. Instructional implications work occurs at the school level as well as at network-wide meetings. We learned that:

- teachers' conceptions of student mathematical thinking ranges from an emphasis on students' procedural understanding of mathematics (for example students having strong number sense) to an emphasis on students' conceptual understanding of mathematics (such as being able to reason, and identify relationships and patterns).
- all of the teachers in this study have participated in at least one of the Network instructional implications meetings (IIMs), and all but one teacher gave a premium value to the IIMs.

- some teachers feel that Two-Pen data are limited in providing them with formative data on their students' mathematical thinking. For some the limitation stems from their belief that the Two-Pen questions are designed to reveal students' computational efficiency, which they see as different from mathematical thinking. For others, the limitations of the Two-Pen problems in revealing students' mathematical thinking is related to the way the actual design and formatting of the problems on the page.

Core Practice: Instructional Planning

The Instructional Planning core practice entails teachers planning instruction based on the formative student data from the math DYOs assessments and discussion of implications of student work. We learned that:

- all but one teacher in this study reported that data from the Two-Pen problems, the context/story problems, or both impacted their instructional planning. Teachers reported two main uses of the DYO data: 1) planning whole class or small group instruction; and (2) grouping students. Several of the teachers also described using the data to select resources or strategies for teaching.

Core Practice: Classroom Instruction

In this core component, teachers implement instruction based on the formative student data from the math DYOs and discussion of implications of student work. To examine classroom instruction we developed five indicators of teacher practices that elicit student mathematical thinking: (1) classroom discourse; (2) open-ended questions/problems, (3) how and why questions, (4) mathematical representation, and (5) use of contextual problems. We learned that:

- all five indicators of teacher practices that elicit student mathematical thinking were observed across the classrooms in this study with varying degree and frequency.
- the most widely observed instructional practice that teachers used to elicit student mathematical thinking was *classroom discourse* (e.g. dialog between teacher and students and among students to share mathematical ideas).
- the least frequently observed instructional practice was teacher questioning that encourages students to engage in mathematical thinking, such as posing open-ended or *how* and *why* questions.

Outcome: Student Demonstrations of Mathematical Thinking

To examine the impact of teacher participation in the PAN 3 Formative Assessment/Instruction Cycle in Math on the students, we developed six indicators of student demonstration of mathematical thinking: (1) reasoning and evidence; (2) observing, conjecturing, and generalizing; (3) making connections, (4) strategizing, (5) communication, and (6) representation. We learned that

- all six indicators of students demonstrating mathematical thinking were observed across the classrooms with varying degree and frequency.
- the most frequently observed indicators of student demonstrations of mathematical thinking in the classroom were *strategizing* (e.g. student using multiple strategies to solve

a multiplication problem) and *reasoning and evidence*, (e.g. students reasoning about the relationship between multiplication and division by proving why a problem was division and not multiplication).

- the least frequently observed indicator of student mathematical thinking was *conjecturing and generalizing*.

Questions for Your Consideration

Our findings raise questions some of which we state here, and others of which are in the text of the report.

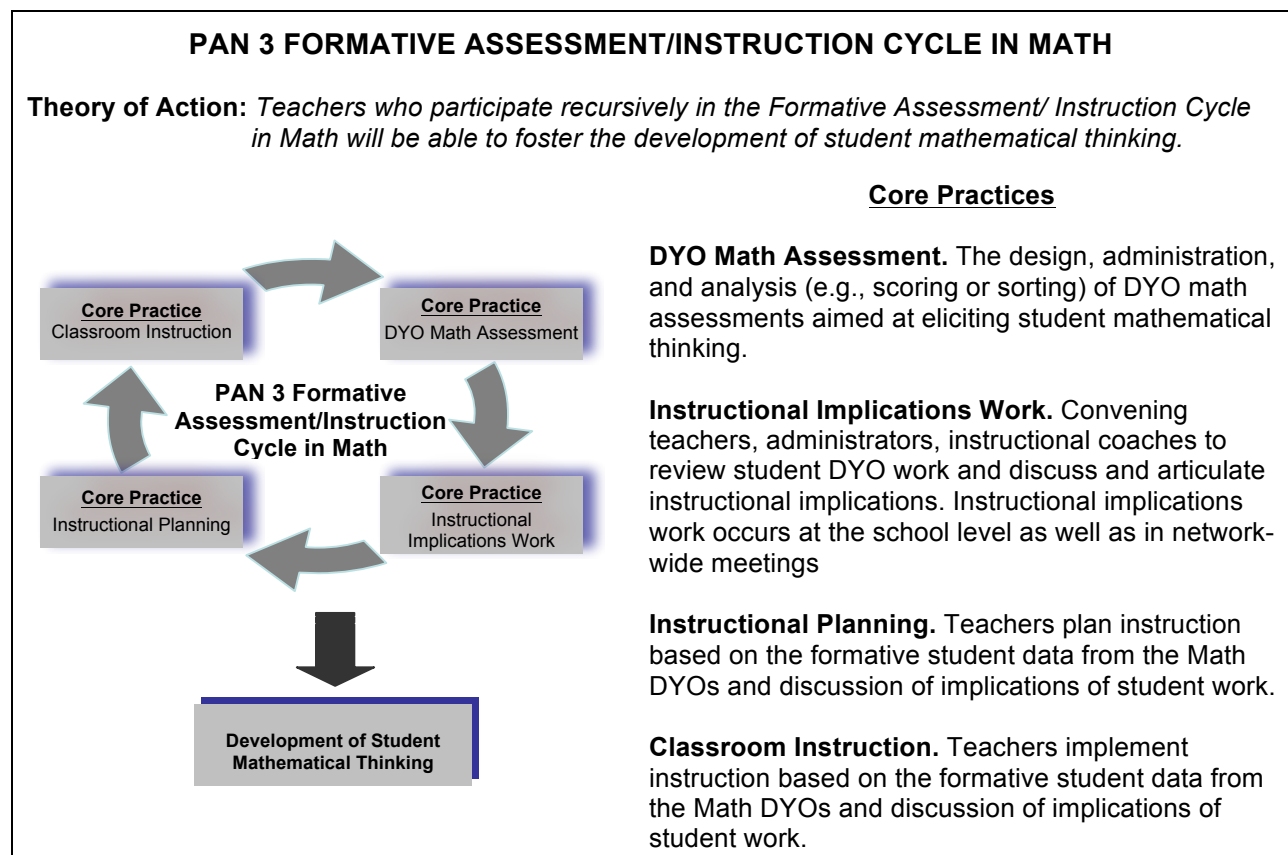
- *Given that those teachers and action researchers who are part of the DYO design team report a positive impact this participation has on their classroom practices, what benefits might result if more teachers or action researchers were involved in or knowledgeable about the DYO assessment design and development process?*
- *Might the Network consider providing a variety of DYO context/story problems from which schools and teachers could chose to administer to students?*
- *How might network-wide professional development further broaden teachers' conceptual underpinnings of elementary mathematics in ways that can support instructional practices that foster the development and growth of students' mathematical thinking?*
- *How can the Two-Pen questions on the DYO assessment be designed to better reveal students' mathematical thinking?*
- *How might network-wide professional development expand teachers' expertise in using questioning systematically to promote students' mathematical thinking?*
- *What teaching practices will promote students' development of the habits of observing, conjecturing, and generalizing in mathematics?*

II. DESCRIPTION OF THE STUDY

1. Introduction

In 2008-09, seven elementary schools in the NYC DOE Empowerment Network 3 invited the National Center for Restructuring Education, Schools, & Teaching (NCREST), Teachers College, Columbia University to collaborate with them on a Practice Area Network 3 (PAN3) study to understand how teacher participation in the Empowerment Network 3 Math Design Your Own formative assessments (DYO) professional community influences or impacts their students' mathematical thinking. NCREST, the DOE, and Empowerment Network 3 principals and staff collaborated on the design and implementation of pilot research study. The goal of this pilot study was to develop and test research instruments and data collection procedures that would best elicit information about the ways in which the Network 3 DYO elementary school math assessments influence teaching and learning in math. Findings from this pilot study would then inform the research design, instrument development, and data collection and analyses procedures for a larger-scale research study.

The PAN 3 Formative Assessment/Instruction Cycle in Math. Based on the findings from the 2008-09 pilot study the ESO 3 Network leaders, principals and staff recognized the need to formally articulate what the PAN 3 network of schools' key math DYO instructional practices and theory of action are, in order to provide a guiding framework for conducting the 2010 PAN 3 DYO research study. The "PAN 3 Formative Assessment/Instruction Cycle in Math" and its component core practices is described below.



2. Research Design

Research Questions. The 2010 study of the PAN 3 Formative Assessment/Instruction Cycle in Math was guided by the following research questions:

What impact does teacher participation in a formative assessment/instruction cycle in math have on:

- a) the design, development, administration and analysis of DYO assessments in math?*
- b) teacher understanding of formative data on students' mathematical thinking and their instructional implications?*
- c) teacher use of formative data on students' mathematical thinking to plan math instruction?*
- d) teacher classroom practices in math instruction?*
- e) the development of third grade students' mathematical thinking?*

Participants. Ten third grade teachers in nine elementary schools from Network 3 participated in the study. Data collection for the study was conducted by ten action researchers from the participating schools with professional development and guidance from NCREST. Each school's action researcher(s) took part in two NCREST-led professional development sessions on action research methodology.

Data Collection. Classroom observations and interviews comprised the data collection methods for the study. (See Appendix A for copies of the data collection instruments used in this study). Action researchers first conducted an initial in-depth teacher interviews to understand teachers' participation in each of core practices of the PAN 3 Formative Assessment/Instruction Cycle in Math and how this participation impacts: (1) teachers' understanding of formative data on students' mathematical thinking and their instructional implications, and (2) teachers' use of formative data on students' mathematical thinking to plan and implement instruction.

The action researchers then conducted one classroom observation focusing on documenting teacher instructional practices that foster mathematical thinking and students' demonstrations of mathematical thinking. A post-observation interview followed the classroom observation to generate more detailed information about instructional practices within the observed lesson.

In addition, the NCREST team collected data on instructional implication work by conducting observations of in-school instructional implication meetings and the Network-wide instructional implication meeting.

Data Analysis and Reporting. To code and analyze classroom observation data, NCREST and the Action researchers developed a set of indicators of teacher instructional practices that elicit student mathematical thinking, as well as indicators of student mathematical thinking. (See Appendix B for detailed description of these indicators). Action researchers coded, analyzed the data and reported findings on how teachers at their schools are engaged in the design, administration, and analysis of the DYO math assessments. In addition, the action researchers reported their findings on the ways in which their teachers engage in teaching practices aimed at eliciting student mathematical thinking, and on evidence of actual student mathematical thinking. NCREST further analyzed and synthesized the action researchers' findings and conducted additional analysis and reporting.

III. BACKGROUND ON MATH IN THE CITY

Learning is a constructive process. Learners need to build the structures for themselves (Fostnot, 2008)

Math in the City is a think tank and national center that provides in-service training and support to K-8 mathematics education. For the past four years, *Math in the City* has worked with a set of New York City schools to create a structure for interim assessments to evaluate the development of student mathematical understanding based on their experience in *Math in the City* classrooms. The nine elementary schools in the NYC DOE Empowerment Network 3 involved in this PAN 3 math DYO study are among the New York City schools that collaborate with *Math in the City* to design and development of their DYO interim assessments in mathematics. This section of the report provides a brief background on *Math in the City* and its approach to teacher instruction in mathematics and the development of student mathematical thinking.

Established in 1995, *Math in the City* is an outgrowth of the collaborative work of Cathy Fostnot, a mathematics education professor at the City College of New York and the Fruedenthal Center in Holland. The Fruedenthal Center's approach to mathematics education is based on the work of Hans Fruedenthal, which he called realistic mathematics education. He viewed mathematics education as a form of guided reinvention (Nickson, 2000) where students develop as mathematical thinkers through context-linked activities. The goal of the combined work of Cathy Fostnot and the Fruedenthal Center was to help "mathematics teachers base their practice on how people learn mathematics, how they come to see the world through a mathematical lens-how they come to *mathematize* their work" (Fostnot and Dolk, 2001, p. xv).

The approach of *Math in the City* to teaching and learning mathematics

supports the argument that when mathematics is taught with realistic contexts, children will build their own ideas and make sense of problems mathematically in their own ways. They will trust in their ability and will at least make attempts to solve difficult problems. In contrast students who are not encouraged to mathematize in their own ways develop a learned helplessness. Math anxiety sets in and when faced with difficult mathematical problems, they give up" (Mathematically Sane, 2010).

One can begin to understand the *Math in the City* viewpoint in how students develop as mathematical thinkers and learners in Fostnot's description of teaching the number line.

We discovered that we couldn't just introduce the number line; we had to create a context to really develop it. The community of young mathematicians needed to "own" the model as a cultural tool, to use Vygotskian language. And then we could use the model to represent their thinking, which is stage 2. Stage 1 is modeling the situation; stage 2 is using it to represent kid's strategies; and stage 3 is where kids will finally take it on and use it as a tool (Johnson, 2008, <http://www.odu.edu/educ/act/journal/index.html>).

In *Math in the City* the student is at the center in the teaching and learning of mathematics. The teacher uses the contextual situations to engage students in mathematical activities in which

students develop mathematical understanding over time. While learning is developmental Fostnot believes that teaching is also developmental. She states,

You become better able to understand what kids are doing by inviting children to engage in investigations and listening to their ideas. There's nothing that you can put in place of that. You just have to start. When I started doing this project 12 years ago there were things I didn't know that I know now about numeracy. I feel I'm still learning how to develop it, and what was behind every kid's idea, and I'm sure I missed tons of wonderful math moments. But by doing it you become better able to facilitate development and maximize the powerful teaching moments. (Johnson, 2008, <http://www.odu.edu/educ/act/journal/index.html>).

Teachers are encouraged to create a classroom culture that fosters student investigation and reasoning with mathematical ideas. The work of *Math in the City* “is driven by the desire to transform classrooms into communities of mathematicians: places where children explore interesting problems and craft solutions, justifications, and proofs of their own making (Fostnot, 2007a, p. 13.). To facilitate construction teachers are encouraged to use “wait time, genuine questioning, the encouragement of classroom dialogue, and the generation of puzzlement” (Fostnot, 2002, p. 41).

The philosophy of *Math in the City* is well reflected in the new common core standards for mathematics presented by National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO) to the nation. There is a common belief that mathematics should be taught for understanding. The assessment process looked at in this study is based on the following statement from the new common core standards:

Asking a student to understand something means asking a teacher to assess whether the student has understood it. But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way that is appropriate to the students' mathematical maturity (2010, p. 4).

Cathy Fostnot has written a series of mathematical books to support the teaching and learning of mathematics based on the philosophical underpinnings of *Math in the City*. For example, *The Big Dinner, Multiplication with the Ratio Table*, uses the context of preparation for a turkey dinner to help students develop understanding of multiplication, “including automatizing the facts, using the ratio table and developing the distributive property with large numbers (Fostnot, 2007, p. 5).

III. FINDINGS

The follow section presents the findings of the 2010 study of PAN 3 Formative Assessment/Instruction Cycle in Math. The findings are organized by each of the four core practices of the cycle, as well as the outcome of the cycle—student demonstration of mathematical thinking. Each core practice is described as well as its corresponding research question(s).

Core Practice: DYO Math Assessment

The DYO Math Assessment core practice of the PAN 3 Formative Assessment/Instruction Cycle in Math includes the design, administration and analysis (e.g. scoring or sorting) of DYO math assessments aimed at eliciting student mathematical thinking. To better under this core practice the following research question was posed:

Research Question 1a): What impact does teacher participation in a formative assessment/instruction cycle in math have on the design, development, administration and analysis of DYO assessments in math?

To investigate this question we asked teachers and action researchers to describe what this core practice looks like in their respective schools as well as the nature and level of their own participation in the DYO assessment (1) design, (2) development (3) administration, and (4) scoring, recording and analysis of data.

1. DYO Assessment Design

Description of the DYO Math Assessment. The ESO 3 DYO math assessments are composed of two sets of assessment problems, intended to provide teachers with formative data on their students' mathematical thinking.

The first set of problems, called the “Two-Pen” is a timed set of bare number problems which gives an indication about students' fluency with either addition and subtraction or multiplication and division. Each problem is designed to be easy to solve if the student understands a particular big idea about the number system or about how the operations work. Students are given three minutes to complete as many problems as they can with one colored pen, then they stop, change for a different colored pen and continue working to complete the set of problems (up to about seven minutes more).

The second set of problems on the assessment is called “context/story problems.” These problems are designed to reveal the types of strategies students are using, the mechanics of how students do computation, and what mathematical ideas are demonstrated in the students' approaches.

Participation in the DYO Math Assessment Design. Teacher and action researcher participation in the design of the DYO assessments varied within and across the schools participating in this study. One teacher (Teacher 1) and three action researchers (Action

Researchers 3, 6, and 7) described participating in the DYO assessment design process. One action researcher and one teacher who are part of the assessment design team explained the reason why the DYO assessments were designed in the way they were:

The DYO assessments came out of displeasure from math teachers with the multiple choice predictives where answers were right or wrong but did not give the teacher any clue as to the mathematical thinking the child was doing. The Interim Assessments designed by the DYO design team were created and field tested to see if they would provide insight into what a child was thinking and how they were solving problems. In many ways, while we always try to focus on the positive and look for what the child knows, many times it is their mistakes that give a clear picture about the child's misconceptions in thinking or the lack of a variety of strategies.(Action Researcher 3)

The DYO is a helpful, useful tool, revealing strategies, not just answers. It's a performance check in an assessment setting. Children don't feel they are in a testing situation, therefore less likely to revert to algorithm. (Teacher 1)

One teacher and one action researcher discussed how designing the DYO assessment has influenced their classroom practice, specifically in connection to developing other assessments and to broadening ideas about teaching math. The action researcher explained that

Designing the DYO assessment has prompted us to write other assessments to be more performance based, like end of unit tests. (Action Researcher 3)

The teacher noted that students are thinking differently about teaching multiplication, information that she has gathered as a result of designing and giving the DYO assessment. She explained that:

students are now looking at the multiplication problems on the 3rd assessment multiplicatively instead of additively. The design of the exam carries over into students' daily work; students are less likely to do things in order; they are more likely to make connections; students do not do as much algorithm work. DYO encourages me to keep doing what we are doing, looking at what kids do and think, and making decisions based on what they do, not what the answer is. (Teacher 1)

Two action researchers (Action Researchers 8 and 10) noted that teachers other than the ones participating in this study were part of the DYO design process. Three action researchers did not include data on their participation in the design process. Although one action researcher did not participate in the design of the DYOs this year, she did have ideas about how they might be revised. The action researcher explained that

I have not participated in the design of the assessments. I have had ideas for revision, however and I have had concerns about the difficulty of some of the problems for students, some with IEPs, who struggle with math skills and number sense. (Action Researcher 8)

Another action researcher explained how all teachers at her school participate in the design process or provide feedback to the design team.

If teachers have a concern about a question on the DYO, we talk about it at our analysis meeting, and then this teacher brings the concern to the design meeting. (Action Researcher 10)

2. Development

Teachers and action researchers described the development of the DYO assessments in different ways. Several participants discussed participating in school-based professional development and providing feedback to the design team in relation to the DYO process and the assessment itself.

I have attended Staff Development sessions aimed at looking at student work. The sessions were intended to give a sense of where students are with their understanding of the concept being taught. (Action Researcher 6)

As a member of the design team I know that we ask for feedback on a regular basis and have used that feedback to make changes to the Interim Assessments.... (Action Researcher 3)

One action researcher was specifically responsible for managing the assessment cycle.

I copy and distribute give memos with timeline for administering the assessments. After the assessment is given, I collect assessments and enter the data onto spreadsheets developed by the DYO design team and return them to teachers with the 2 pen assessment scored on the spreadsheet (Action Researcher 3)

In one case, the administrators or the school math coach were central in supporting teachers and providing feedback to the design team during this development phase.

The two-pen was introduced by our principal before we had [Ms. S.] as a math coach. We were fairly confused about how to administer the assessment and probably made many mistakes. Now, we have bi-weekly math meetings and [Ms. S] reminds us of the schedule and by when we need to administer the assessments. Recently we have been helping [Ms. S.] input the data from the assessments into Filemaker Pro program at school. The second set of data, Ms. S. input for us. This way, we can look at student progress over time. (Action Researcher 8)

Questions for Your Consideration

- *What benefits might result if more teachers or action researchers were involved in or knowledgeable about the assessment design and development process?*
- *What formal procedures are in place within the Network for schools and teachers to provide feedback on the DYO assessment design?*

3. DYO Assessment Administration

In each school participating in this study, teachers were responsible for administering the DYO assessments. In two schools, School 3 and School 8, action researchers reported that teachers are given support to administer the assessments as the need arose. This support was generally provided by a math coach. Action Researcher 10 noted specific directions which teachers in her school gave to students before administering the assessments:

Before giving the Two-Pen, teachers tell their students to solve the problems first which they think are easier; teachers let their students know that it is okay to solve the problems in any order. Before giving the story problems, teachers tell the students to show their mathematical thinking. (Action Researcher 10)

Action Researcher 3, who is a member of the design team, describes her involvement in the administration of the DYO at her school:

I make sure the copies are made and I craft a memo to teachers about the timeline for administering the [DYO assessments] and my availability for assisting them in administration if they feel the need. I provide professional development around how to administer them. (Action Researcher 3)

4. Scoring, Recording and Analysis of Student Assessment Data

Action researchers reported varying ways in which the DYO assessments were scored, recorded and analyzed after they were administered by teachers.

Two-Pen. The responsibility for scoring and recording data from the Two-Pen assessment fell to different individuals. In School 2, a “data person” (a classroom teacher) collected the data from the Two-Pen assessment administered by the teachers and entered them onto a spreadsheet. In School 3, the math coach scored the Two-Pen, entering the data onto the DYO Design Team’s spreadsheet. In the other schools, teachers were responsible for scoring and entering data from the Two-Pen onto the spreadsheets themselves. Only School 5 reported that it did not use the Network database for scoring the assessments, but instead had developed its own scoring methods for both the Two-Pen and the context problems. The action researcher at this school commented:

Teachers note if answers are correct, if correct operations are used, and what strategies are used on the score sheets. The sheets are kept on file by the Assistant Principal. In addition, a cover letter with a summary of the assessment is sent home along with the actual assessment for parent signatures. The assessments are kept in the student’s files. (Action Researcher 5)

Context/Story Problems. The ways in which teachers scored and recorded data from the context/story problems varied across schools. Action Researcher 3 noted that teachers used their own methods for scoring and storing the data from the story problems. Action Researcher 10 commented that teachers scored the story problems by rating each student’s efficiency on a scale of 1-4 as well as scoring the number of problems answered correctly. Action Researcher 4

mentioned that the context problems were sorted on the rubric. In School 1, teachers sorted them based on strategies being used. Then, a learning landscape spectrum is designed for each class based on what the students were doing. This learning landscape generally ranged from less sophisticated for inaccurate strategies to more sophisticated, accurate, and efficient strategies. The work at the upper elementary grades also filtered down to the lower grades in that school with the hope that:

This work will help make our math curriculum more cohesive as a whole school and that teachers will develop a sense of a child's math learning trajectory beyond what just occurs in any one particular grade or year for a child. This work has come out of our participation in the Math DYO. (Action Researcher 1)

One teacher questioned the relevance of the context/story problem questions on the assessment itself, depending on where she was in the curriculum she was using. She commented:

With the extended response, too many of them can't do it at the beginning of the year and so it doesn't really impact my teaching because I already know most won't understand it and I am about to teach into these ideas soon. I think addition might be more informative because most children can't solve the subtraction problem as comparison early in the year. (Teacher 2)

She contrasted that to the Two-Pen as she said, “with the Two-Pen you can see what relationships they have, they are using, and start from there.” Overall, she felt that the assessments were at their best when they coincided with the content of the curriculum at a particular point in time as she asserted that, “the multiplication was perfect because it was right in the beginning of the unit and I could use it in planning my daily work more easily.”

Question for Your Consideration

- *Might the Network consider providing a variety of DYO context/story problems from which schools and teachers could chose to administer to students?*

Core Practice: Instructional Implications

The Instructional Implications core practice of the PAN 3 Formative Assessment/Instruction Cycle in Math involves convening teachers, administrators, and instructional coaches to review student DYOW work and discuss and articulate instructional implications. Instructional implication work occurs at the school level as well as at Network-wide meetings. To better understand this core practice, the following research question was posed:

Research Question 1b): What impact does teacher participation in a formative assessment/instruction cycle in math have on teacher understanding of formative data on students' mathematical thinking and their instructional implications?

To investigate this question we asked teachers about (1) their understanding of student mathematical thinking; and (2) how data from the DYOW assessments impact their understanding of student mathematical thinking. Teachers also reported on their participation in the network-wide instructional implications meetings (IIMs) and the impact they believed that the IIMs had on their understanding of formative data and their instructional implications. And lastly, the NCREST team observed two school-based instructional implications to better understand how information from the IIM is turned key at the schools

1. Teacher Understanding of Students' Mathematical Thinking

There was variation in teachers' understanding of student mathematical thinking. Some teachers' conceptions of student mathematical thinking emphasized strategic flexibility:

...mathematical thinking means understanding broad concepts – big ideas – like understanding what multiplication is. If you do, and those ideas are solid, then you are able to manipulate it so you can draw from a variety of strategies to solve a multiplication problem. This shows that you really understand the concept. (Teacher 5)

In third grade, we give kids... problems and allow them to work through it. We want more than one way to solve a problem and we lead them into new ways. (Teacher 8)

Other teachers noted students' conceptual understanding of mathematics, and emphasized student problem solving, reasoning, and identifying patterns and relationships:

Problematic situations, just having to figure something out, using all these different things that you know to figure out something else—that wonderful feeling when you figure it out when you didn't know before. (Teacher 2)

It means to be able to see the relationship between numbers, using things like the distributive property. I personally never thought about that as a kid or even when I first became a teacher. So to have that fluid and flexible way of using numbers, of adding numbers, subtracting numbers, thinking about numbers, imagining numbers. (Teacher 4)

[It] means kids can see patterns in everyday life; they can estimate things, you know, the spatial relations I try to get my kids to see that mathematics in their everyday lives as much as possible and I view mathematical thinking as being able to recognize that. (Teacher 3)

Mathematical thinking is a way of understanding the world through logic, reasoning, looking at relationships, [and] looking for patterns. It's also a way of communicating your thinking through the use of symbols [and] visuals. (Teacher 1)

Mathematical thinking is more of a process where anyone shows their ability to reason, to problem solve, to make connections between the concepts and ideas they know and build relationships between those concepts... [for example], making connections between multiplication and division. (Teacher 9)

Questions for Your Consideration

- *How might network-wide professional development further broaden teachers' conceptual underpinnings of elementary mathematics in ways that can support instructional practices that foster the development and growth of students' mathematical thinking?*

2. Impact of the DYO Assessment Data on Teachers' Understanding of Student Mathematical Thinking

Regardless of their conceptions of student mathematical thinking, most teachers in this study reported that the DYO assessments helped revealed to them how their students were thinking mathematically. Some of these teachers noted that the DYO overall (both the two-pen questions or the story/context problems) impacted how they viewed their students' mathematical thinking,

[The DYO] has helped me to stay closer in touch with the ways in which students are thinking and how to advance their mathematical thinking. It has also helped me to design math units that are connected to context in the real world. (Teacher 6)

One benefit, a real plus, is to be able to view math solutions by children on a continuum more carefully and more personally. The DYO puts a lot of emphasis and importance on the thinking. The assessments really look at and really allow us to study a child. I have gotten the idea of analyzing the minutiae of what they're doing. (Teacher 8)

The DYO is a helpful, useful tool, revealing strategies, not just answers. It's not just a test, but a performance check-in in an assessment setting. The DYO assessment encourages me to keep doing what we are doing, looking at what kids do and think, and making decisions based on what they do, not what the answer is. (Teacher 4)

Two Pen Questions. The Two-Pen questions of the DY0 assessments were, for some teachers, effective in revealing their students' mathematical thinking.

I could see overwhelmingly that more kids were able to do the problems in the first three minutes and they really used the turn around facts and the doubling. This time they really could see the adding with multiples of ten. (Teacher 3)

[The Two-Pen] helps me see who has good mental math strategies and who sees the relationship between numbers...[it] helped me notice how hard it was for my students to see the relationships between numbers. (Teacher 5)

The Two-Pen has a lot of related problems... that if they get it immediately, I know that they are using a basic fact to help them. If they take a long time to figure it out, I know they are not seeing the connection to a fact that would help them solve the problem more efficiently. (Teacher 6)

Other teachers asserted that the Two-Pen problems are limited in revealing student mathematical thinking. For Teacher 9, the limitation of these problems on the DY0 relate to her conceptions of student mathematical thinking:

I still believe the Two-Pen is about automaticity not thinking unless the teacher takes the problems into number strings and explicitly teaches to make connections and teach strategies. The DY0 alone does not do that.

I see "strategy" as different from mathematical thinking. Mathematical thinking is a process... A kid can solve $60+80$ is 140. Just because $6+8$ is 14 does not show thinking, just a shortcut. If you can express your strategy to someone else of how it works and that person understands how it works and why, then perhaps some thinking is going on. (Teacher 9)

For Teacher 7, the limitations of the Two-Pen for revealing mathematical thinking can be attributed in part to the format of the problems:

Two-Pen shows us fluency. Sometimes you can see thinking ... if they made tally marks on the page or showed their strategy, and sometimes if they got or did not get the related problems, though the format of this test doesn't really facilitate kids using the related problems. I have noticed kids who would normally use related problems not use them on the Two-Pen. (Teacher 7)

Teacher 10 was disappointed in her students' performance on the Two-Pen and speculated that the timing and testing pressure compromised the assessments formative value.

Two-Pen is more mental math. I was disappointed with their overall performance on 402-399... For some of them, it was an "aha moment." I wanted them to see how close together those numbers were and what a short distance they're actually looking at. Instead, almost none of them got that one. Maybe with the time, and test pressure, they couldn't apply the same logic. (Teacher 10)

Context/Story Problem Questions. In general, most teachers reported that the context/story problems revealed their students' choice and efficiency in the use of strategies to solve the problems.

[The] open ended [problem] shows strategizing ... if student understanding is superficial or deep, we can tell by the strategies they use. It promotes problem-solving. [Students] need to figure out the context of the problem and figure out what it is they should be solving. (Teacher 9)

...the work here will reveal the students understanding of operations, their understanding of place value and will show how efficient they are with these particular numbers. The subsequent problems are related to the earlier problem and will show if students have the insight to use the work they have already done to help them. (Teacher 6)

From the work I can see who has efficient methods and who can see an easy way to solve a problem. And, I can see who is less able, and using less efficient ways. I look at kids' work and notice scribbling from which I can tell who is working through something and not giving up. (Teacher 8)

The story problems – can they understand what they are being asked to do, that it requires them to use the data at the top of the page, use the correct operation, is the strategy efficient, flexible? (Teacher 7)

...with the word problem, you can see ..., do they conceptually understand a multiplicative situation? Some kids will add. For those who do understand it as a multiplicative situation, you can see what strategy they are using and see how to push them forward to a more efficient strategy. You can see that they might have some mistake in their procedure that you can help them with, even though they are understanding conceptually. (Teacher 2)

Questions for Your Consideration

- *How can the Two-Pen questions on the DYO assessment be designed to better reveal students' mathematical thinking?*

3. Impact of the IIMs on Teachers' Understanding of Student Mathematical Thinking and Its Instructional Implications

After the DYO are administered, the Network holds Instructional Implications Meetings (IIMs) so that teachers have the opportunity to look closely at their students' DYO work and understand their instructional implications. As one action researcher, who is also a member of the DYO design team, noted:

We, as a design team, feel strongly about the assessments being followed by the Implications Meetings so that teachers are able to discuss and really analyze their student work and return to their schools with some concrete ideas for next steps for their students. (Action Researcher 3)

Of the ten teachers involved in the study all have attended at least one network IIM, and two teachers have co-facilitated IIM sessions. With the exception of one teacher, all teachers reported that the IIMs have had an impact on their instruction.

Teacher 1, who has served as a facilitator at the IIMs, noted “[I] understand and believe even more how important it is to look at authentic assessment and it makes me want to create an environment where this kind of work is fostered.” Teacher 8 noted that at one IIM he was introduced to a different approach to teaching multiplication which pushed his thinking from using skip counting into regrouping numbers into pairs or clusters and then doubling or using addition to solve problems more efficiently.

Teacher 2 described how she had a greater understanding of children’s mathematical thinking as a result of the IIMs:

Going to the implications meetings seems to be the most impactful. ... The share part of the meeting, sorting work and trying to understand what the student is doing, ..., why did they do what they did, what is the sense they are making... I have gotten better at seeing a child’s thinking in their work and asking questions based on that work and what they say. (Teacher 2)

For Teacher 9, the IIM changed the way she saw the DYO, akin to an epiphany:

In the beginning, I did not see the reason or purpose for administering the DYO. I felt like we were doing it because we were told to do so. This changed when I went to the last IIM and I saw that I could use this to help me teach and learned who designed them. I like that I saw that people who designed them were actual teachers. I used to think that they were designed by people who were no longer in the classroom....After the IIM I saw how different the kangaroo and juice problem were and their purpose, I saw how a teacher looking at them carefully would be able to get information from it. It helped me understand the need to develop new number strings and other things to help develop their concepts to give them a toolbox of things to use. (Teacher 9)

Teacher 10 and Teacher 5 specifically noted how the IIMs helped them teach arrays:

I was ... able to recognize spill over from this lesson into an array lesson conducted after this lesson, something I might not have realized was a close connection had I not been to the [IIM]. (Teacher 10)

The meetings have definitely helped me.... At the last IIM we talked about arrays and I realized that I needed to return to the array model to help [my students] move away from using repeated addition. The meeting clarified that for me. (Teacher 5)

Teacher 5 added that she would like more time to meet with colleagues to talk about student work:

I think it’s important for the DYO people to inform principals who participate in the DYO that it’s really important to have time to be built into teachers’ schedules to allow them to meet with each other to analyze the assessments, design ways to integrate the results into our teaching, and to score the assessments. (Teacher 5)

One teacher, Teacher #7, in contrast with the other teachers in this study, felt that the IIM did not have an impact on her practice:

The IIM is feeling less relevant and more redundant and repetitive. We need to mix it up...The IIM has not directly influenced the way I plan or group kids. The kids move on from where they are so quickly that within a month it's already dated information.... I don't think we always do enough 'hard math' at the IIM—the meetings have not pushed my own math thinking in a way that helps me in my practice. (Teacher 7)

Questions for Your Consideration

- *Given that most teachers reported a positive impact of the IIMs on their instruction, how could increasing their frequency be beneficial to the teachers? How frequent might be optimal?*
- *What structures can schools and the network put in place to ensure that all teachers value and benefit from the IIMs?*
- *How might principals increase time for teachers to assess student work and discuss with colleagues?*

4. School-Based Instructional Implication Sessions

Following the Network-wide IIMs, several action researchers noted that their schools conducted school-based instructional implications sessions to turnkey the information from those Network meetings to grade level teachers. Sometimes this was a morning meeting while for others schools it was at lunchtime or at other times in the day.

One NCREST researcher had the opportunity to visit two schools, School 3 and School 5, to observe how teachers turn-keyed information from the March 2010 Network IIM to their grade level teams. What follows is a summary of how the two meetings proceeded.

School 3. The meeting at School 3 took place early in the morning before students had arrived. Five grade three teachers and the math coach were present initially. The principal arrived during the last ten minutes of the meeting. Teacher 3, who had attended the IIM, led the half-hour meeting. She began with the stamp problems from the most recent assessment. She handed out the rubric given at the IIM and told her teachers that she chose to number the stages 1-5 for herself right on the rubric. She proceeded to explain the rubric. She then asked the grade level teachers to work in pairs to sort their students' stamp problems based on the rubric. She suggested that teachers use Post-its to sort into the groups. One teacher commented that at first she had difficulty seeing the numbers on the stamps. The math coach responded that this would be addressed at the next design team meeting. Another teacher suggested giving the assessment over two days.

As the pairs were sorting their students' work, questions which emerged included:

- "If kids are using multiple strategies, do you look at the lesser of the strategies?;

- “Would it be a separate group if kids didn’t recognize turn around facts? [referring to the last stamp problem];
- “What if they used a strategy, but got the answer wrong? [juice box problem];
- “What if children skip counted incorrectly? Where to put them?”

The questions were never brought up to the whole group to discuss since they arose during the conversations among the dyads. At one point, however, Teacher 3 said that if children used a wide variety of strategies and got the answer correctly to put them in the highest group. From the assessments, another teacher said that she needed to come up with context work problems (e.g., bring in real juice boxes and break down the problem).

After teachers finished sorting the assessments according to the rubric, Teacher 3 told them about the games that were introduced during the IIM (e.g., The Array Game, Who Will Say?). She also went over Quick Images. She ended by asking if anyone had any questions. Teachers said students had trouble getting past 99 with subtraction. Some suggestions were: Give a lot of strings, mini-lessons, and quick 10 minute warm-ups. The meeting adjourned after a half hour.

School 5. The meeting at School 5 took place in the afternoon of the same day as the School 3 visit. Nine grade three teachers and the principal were involved in this pizza lunch meeting. In contrast to School 3, the principal at School 5 was an active participant in the discussion and assessment of data. Teacher 5 turn-keyed information from the IIM. The teacher at School 5 did not initially show the teachers the rubric for the stamp problem. Instead, she instructed pairs of teachers to look at the stamp problems and sort by strategies they noticed children used. They were free to sort according to what they were noticing. During their sorts, they did not seem to have the same type of questions or debates about where to place students as at School 3. Half-way through the 35 minute meeting, Teacher 5 wrote on the whiteboard:

Drawing
Repeated +
Doubling/grouping +
Skip counting
Partial products
Doubling x
Memorized
Easier fact, then + or –

She stopped the teachers briefly to then show them the rubric from the IIM, but discussed how she used it differently. She listed her class’ strategies on the whiteboard. On her class list, she made note of every strategy each student used. This gave her significant information about where she wanted her class to go. She showed the other teachers her class list. Next, she had teachers take a few of their students’ assessments and record all the strategies they noticed each student using. Some questions which emerged were: What if they used “memorized” and “got it wrong?” “Are tallies drawing?”

All teachers saw that their students used multiple strategies. Further discussion about the stamp problem and its findings ended so that Teacher 5 could have time to show and briefly explain the packet of games handed out at the IIM, as the teacher from School 3 had also done.

Core Practice: Instructional Planning

The Instructional Planning core practice of the PAN 3 Formative Assessment/Instruction Cycle in Math entails teachers planning instruction based on the formative student data from the math DYOs and discussions from implications work. To better understand this core practice the following research question was posed:

Research Question 1c): What impact does teacher participation in a formative assessment/instruction cycle in math have on teacher use of formative data on students' mathematical thinking to plan instruction?

To investigate this question we asked teachers to describe the ways in which they use the math DYO data to plan instruction.

1. Teacher Use of Formative Data on Students' Mathematical Thinking to Plan Instruction

The DYO math assessments provided teachers with a vehicle for looking at how students in their classes were faring mathematically. How teachers used that information to structure their classroom practices varied. Nine of the ten teachers who participated in the study reported that they altered classroom practices as a result of the DYO assessments. When asked about their use of formative DYO assessment data on students' mathematical thinking to plan math instruction, the majority of teachers described two main uses: 1) planning whole class or small group instruction; and (2) grouping students. Several of the teachers also described using formative data on students' mathematical thinking to select resources or strategies for teaching. One teacher in this study noted that the DYO data had minimal influence on his practice and did not impact his instructional planning.

Planning Whole Class or Small Group Instruction. Most teachers described various ways that formative data on students' mathematical thinking informs their decision making about instruction:

For the Two-Pen [of the DYO assessment], I score them on a spreadsheet and look for patterns within the class. I see if there are certain kinds of problems that a lot of kids are getting or not getting. Depending on how those groups come out, I do whole group lessons or small group lessons. (Teacher 1)

I try to get a sense of the whole class. If there is something that the majority of the class is struggling with I try to address with the whole class (Teacher 2)

The Two-Pen in the beginning showed me a lot because there are quite a few kids who are having trouble counting on from 99 and I know I need to go back and address that. (Teacher 3)

We sorted [students' DYO assessments] along the rubric and identified those [students] that need a "shove" towards multiplicative thinking. And after sorting, these children became part of a small group for intervention work with our inquiry team. (Teacher 4)
The data make me think about what each kid needed to take them to the next level of thinking or master the level they are at right now. I saw that kids were at very different places and I made the lesson so that it was accessible and useful for the majority of students and that everyone could gain from it at his/her own level. (Teacher 1)

Scoring the [DYO] assessments helped me understand what my kids are doing, and how they're using repeated addition as a strategy instead of more efficient strategies. I could see that even my stronger kids were relying on this too much. So I knew that I needed to address this with the whole class. (Teacher 5)

Grouping Students. Teachers also stated that they used DYO data to make decisions about grouping students within their classes:

I make small groups based on the data I gather to focus on what I see they need to work on. (Teacher 5)

I did their seating based on the DYO. (Teacher 2)

The higher level group is students that I've seen work really well with their multiplication facts. They know them automatically. They're able to play with numbers. And also on the Interim Assessment I noticed that they were able to come up with the right answer and they used strategies to figure it out. With the lowest group, those were the kids who really didn't use a strategy. Maybe they were counting by ones and so I wanted to play that game with them so they could force themselves to count by 2s and 3s and 4s. The middle groups were groups that maybe I noticed were getting their five facts right and their two facts and four right but needed to work on the 7s 8s and 9s. So I wanted them to practice that with the array game. On the rubric it says something about counting by ones and using tedious strategies; the low group was those kids when I looked at the interim assessment that fell in those areas of the rubric. (Teacher 3)

The [DYO] results did help me group students at the beginning of this unit. I was able to pair students at similar places along the multiplication rubric. And the work encourages me to continue multiplication strings because it is clear to me that students have done well on addition and subtraction after all our string work. I have used the data to identify groups for morning program and intervention. (Teacher 4)

Selecting Resources or Strategies for Teaching. Several teachers also discussed how formative data on students' mathematical thinking informed their decision making on the selection of instructional resources and strategies:

The data show me on what level of math thinking my students are operating and in general where they are. It helps me to see who needs more work and who needs something more challenging. It gives me an idea how I can support students with more efficient problem solving strategies and how I can help them use existing knowledge to solve related problem [for example) 200-3, then 201-3. (Teacher 6)

I create math strings to deal with specific issues like subtracting numbers that are close together versus numbers that are far away from each other, or looking at doubles. I also try to make more context problems that will help them continue to do good problem solving work for improving efficiency. (Teacher 5)

One way I use the data is to teach number strings or to develop efficient strategies. With the two-pen it's all about the strategies they use, number strings, getting them to develop efficient use of strategies. The data has shifted my math routines in the mornings. Now I do short investigations with smaller numbers for children to work on that help them think about concepts that were presented in [open ended problems] and also in the two-pen (Teacher 8)

The two pen in the beginning showed me a lot because there are quite a few kids who are having trouble counting on from 99 and I know I need to go back and address that. ...I'm becoming more comfortable with number strings so I think that would be the easiest way to start. I could see them using some sort of manipulative here to balance things out. I might even use weights or something for some of these problems. (Teacher 3)

The data has helped me to design math units that are connected to context in the real world. I incorporate a lot of related problems into the lessons so that kids can see the connection between problems. For example, I showed a set of 2-cent stamps (8) and kids had to find the value. We did the same with 4-cent stamps, the same amount. Students then had to figure out why the answer had doubled. This lesson built on the grocery store unit that we did. We used this as a jump off for finding sets of arrays throughout the school. (Teacher 6)

The reason I planned this lesson... was because according to the DYO data, my kids were not fluent or efficient in their understanding of multiplication. A lot of them were relying on skip counting and inaccurately. And it was inefficient. I wanted to teach them a way to use smaller facts to get bigger facts. (Teacher 1)

In the DYO I noticed that there were a number of students who were using skip counting as their only strategy.... I would focus on a lesson around creating groups and using groups as a way to count to find the total. I would not focus on factors. (Teacher 9)

In one case, not only did the DYO formative assessments alter instructional practices in the classroom, it extended further to school-wide practices as it impacted the work of the school's Inquiry Team. This teacher explained:

When we sorted them here at school [the 3rd assessments] we looked more closely at multiplication. We sorted them along the rubric and identified those that need a 'shove' towards multiplicative thinking. And after sorting, those children became part of a small group for intervention work with our inquiry team. (Teacher 4)

All teachers in this study, with the exception of Teacher 7 described the ways they used formative data on students' mathematical thinking to inform their decisions on the planning of math instruction. Teacher 7 stated that the DYO, "is just a window – I don't really use it to go back and re-teach, just to take the temperature of the class."

Question for Your Consideration

- *What steps might the network take to ensure that all teachers participating in the Math DYO initiative use the data on students' mathematical thinking to inform their decisions on delivering instruction in the areas of grouping students, instructional strategies, they and the resources?*

Core Practice: Classroom Instruction

The Classroom Instruction core practice of the PAN 3 Formative Assessment/Instruction Cycle in Math entails teachers implementing instruction based on the formative student data from the math DYOs and discussions of implications of student work. To better understand this core practice the following research question was posed:

Research Question 1d): What impact does teacher participation in a formative assessment/instruction cycle in math have on teacher classroom practices in math instruction?

To investigate this question, action researchers conducted one classroom observation of the teachers participating in the study. (See Appendix C for the action researchers' summaries of observed lessons). Action researcher then coded the data along the five indicators of teacher instructional practices that aim to elicit student mathematical thinking: (1) classroom discourse; (2) open-ended questions/problems; (3) "how" and "why" questions; (4) mathematical representation; and (5) use of contextual problems. A few action researchers observed instructional practice that they believed did not correspond with these five indicators and coded these practices as (6) other.

1. Instructional Practices That Elicit Student Mathematical Thinking

Action researchers observed all five indicators of teacher instructional practices that elicit student mathematical thinking with varying degree and frequency. The most widely observed instructional practice that teachers used to elicit student mathematical thinking was *classroom discourse*. The least frequently observed instructional practice was teacher questioning that encourages students to engage in mathematical thinking, such as posing open-ended or *how* and *why* questions.

What follows are highlights from the classroom observations of each of the teacher instructional practices that elicit student mathematical.

Classroom Discourse. Classroom discourse consists of any of a variety of exchanges in which ideas about mathematics are communicated within a classroom. As an instructional strategy, this may include dialogue between teacher and students, dialogue among students, presentations, use of images or diagrams, or any of a host of communicative methods to share mathematical ideas.

In this study, classroom discourse was the practice most frequently observed instructional practice by the action researchers. The most commonly observed strategies to foster classroom discourse were: (1) Teacher communicates expectations for students to explain their thinking process (es) and defend them to one another; (2) Teacher poses a problem to students and allows them to find a solution either on their own or in collaboration; and (3) Teacher allows for student-driven discourse in groups.

Classroom discourse involves teachers communicating their expectations to students, such as Teacher 5's and Teacher 10's communication to students:

That's great, [student name]. But when you guys play, try using two smaller parts first instead of three. So your goal for this game is to pull a big array randomly and then you're going to try to find your smaller arrays to cover the big one, okay? I'm sending you off in pairs to play. If you understand how to play I'll let you choose your partner. If have any questions or if you need to practice this with me, stay on the rug. Don't forget to take your math journal with you to write down the equation for the arrays.

Remember [student name], you need to use parentheses if you're going to write the equation this way, otherwise it isn't really correct. And don't forget to add the plus sign between each one. Can you do that for me now? (Teacher 10)

Classroom discourse also entails having students defend their mathematical thinking to peers and teachers as the following example showcases:

As children prepared posters of their mathematical thinking about various dimensions of boxes of 10 and multiples of 10, teacher asks students to 'focus on the information. Can it speak for itself on your poster? Is your big idea true? Focus on math and writing, not art. It should be clear and precise?...What big idea can you convince people of?' (Action Researcher 4)

In this example, students were expected to justify their answers in verbal and non-verbal formats.

In each observed lesson, classroom discourse was fostered by small and large group discussions and activities, as evidenced by the summaries of the observed lessons in Appendix C. Most lessons opened with teachers explaining the math tasks to students in a whole group format. Next, students were generally asked to complete tasks in small groups. Finally, many teachers brought closure to the lessons back in a large group setting.

Open-Ended Questions/Problems. Asking open-ended questions is a key strategy teachers use to stimulate mathematical thinking. Open-ended questions can embody the demand for student exploration, sense-making, application of ideas, extension of ideas, construction of new ideas, and struggling with the unknown. The teacher provides for multiply entry points that can lead to one or multiple solutions. In classroom observations, action researchers found a range of open-ended questions being asked by teachers.

Generally, teachers in this study began the observed lessons with an open-ended question or problem that became the focal point for the lesson. In one lesson, Teacher 9 began with the following open-ended problem:

Joaquin needs to organize his chocolates and he needs us to help him think about how to organize 12 chocolates. You are going to work on showing how to show and develop all of the boxes for 12. Then you are going to think about how you know that you have found all the ways. (Teacher 9)

This teacher further commented on the importance of open-ended questions/problems:

We began doing these kinds of open units of study. They are open and allow kids to explore problems with others, to problem-solve with others. The open investigations help promote math thinking. The congress or share is a helpful way to promote math thinking, process allows students to collectively learn from presenters. Students discuss different approaches and strategies. Because the problems are open, in the sense that you present problems in context and they go off and try to find a way to solve it, they're forced to use what they know and build from there. Mathematicians learn in groups instead of individually by understanding each others' work. (Teacher 9)

The theme of multiple entry points via open-ended questions was highlighted by many of the teachers. Teacher 2 noted that she assesses mathematical thinking using open-ended questions:

I go around and ask them to explain what they are working on, and how they are figuring out what they are doing. (Teacher 2)

One action researcher noted some of her observations about opened-ended activity in the classroom. The action researcher wrote:

The task the children were engaged with during the lesson observation also had multiple entry points—the work was to discover a way to figure out how many marbles a mythical child would have after 30 days. During the warm-up the teacher asked many open-ended questions: 'E, what do you think?' and 'I want to hear your reasoning, why do you think it is true or false?' and an invitation to share thinking: 'Tell me more.' As the teacher circulated, she asked many open-ended questions. Some were to help students get started: 'What do you think you can do to show your work?' 'How are you going to show me that?' Others asked students to reflect on what they were already doing: 'How are you going to know when you have enough 2s?' and 'Do you think you can draw a picture to keep track?' At other times, the open-ended questions were an extension: 'How many marbles would she have on the 24th day. Would you be able to figure that out?' and 'What if I asked you how many marbles she has on the 19th day?' (Action Researcher 2)

Action Researcher 4 noted how the teacher she observed focused on the students' thought processes rather than a fixed answer:

During my time in the classroom I did not hear the teacher or students ask 'What's the answer?' Although an answer or conclusion was arrived at, the work was about thinking, process, evaluation, strategy, modeling, communication. Even when specific multiplication problems were posed they were written on the board without equal signs, leaving them open to discussion, not immediate answers. When one student instantly answered '38' for 12×19 the question 'What do you think class?' prompted others to share their strategies and not dwell on the incorrect 38. The open-ended questions allowed every student an entry into the problems and promoted a range of big ideas from 'All the numbers we make with the 2×5 arrays are multiples of 10' to 'When you double one factor, the product doubles.' (Action Researcher 4)

“How” and “Why” Questions. Asking “how” and “why” questions give students an opportunity to describe what they did and why they did it. The students have to be able to give reasons for the choices they made. Reasoning through problems and offering evidence in support of ideas are essential components of mathematical thinking.

More so than some of the other instructional practices, teachers in the study utilized the “how” and “why” questions to have students explain their thinking or procedures and provide evidence. As Action Researcher 8 noted, “[the teacher] is constantly pushing and challenging children to push themselves to think differently, to examine their own assumptions, and to question each other’s ideas.”

In the interview, Teacher 9 talked about the importance of questioning:

Mathematical thinking is a process; kids won’t develop this if teachers don’t get it. Teachers won’t know the questions to ask to get certain responses from kids if they don’t understand the concepts. Teachers need to ask questions that help them realize and use what they know, questions that help them build upon what they know, use what they know, looking at others’ work. Questions should touch upon the various levels of math thinking, the steps in the process of mathematical thinking. (Teacher 9)

The action researcher who observed Teacher 9 noted how students’ responses informed her instructional strategy:

When conferring with partnerships, the teacher changed her questioning to clarify and extend the thinking that partnerships were doing. She refocuses them. There is a 2x5 box and she asks a student how many are in the box. He says 10 and she asks if it would fit 12. The student says no so the teacher asks, ‘What do you need to add to the box to make it fit 12?’ He says, ‘Add a square’ and she clarifies and asks, ‘one little square?’ and he says, ‘No row.’ The teacher sets up partnerships. Students in partnerships are expected to ask questions of each other. (Action Researcher 9)

Another action researcher wrote about how a teacher used open-ended questions to elicit students’ mathematical reasoning:

The teacher continually asks students to explain how and why they did something, and to connect what they did to the logic of mathematics. In the math warm up she says, ‘I want to hear your reasoning, why you think it is true or false.’ She tells the class, ‘We have to convince N or she has to convince us.’ She asks, ‘Why did he say 3×30 is 90 and then $90 + 30 = 120$. Why was this the equation?’ In the post interview she specifically references liking the share at the end because the question about the meaning of the numbers created a discussion where students were ‘trying to understand someone else’s work and add on to each other’s thinking.’ (Action Researcher 2)

Mathematical Representation. Mathematical representation refers to multiple ways in which mathematical concepts are presented. Because one specific idea can often be presented in many

different forms – in number, in the English language, in diagram – it is important to the development and expression of students' mathematical thinking for them to understand how they can create mathematical representations to express their mathematical ideas. The observed lessons showed a range of uses of mathematical representations, including both student and teacher representations. Some of the ways in which information was represented was through the use of different colors, grid paper, words and symbols, posters, models, diagrams, and manipulatives.

Teacher 5 stressed the importance of representing mathematical equations using correct notation. She seemed to value the importance of conveying this information to her students as the action researcher described:

The teacher clearly values the importance of correct representation of the array models ('Big Plates' and 'Side Dishes') in this activity. She demonstrates two different ways to write the problems herself, one in a vertical format, the other in a horizontal format (using parentheses).

Examples:

$$\begin{array}{r} 2 \times 6 = 12 \\ 5 \times 6 = 30 \\ \hline 7 \times 6 = 42 \end{array} \qquad \begin{array}{r} 7 \times 6 = (2 \times 6) + (5 \times 6) \\ 12 \quad + \quad 30 \quad = \quad 42 \end{array}$$

The teacher also insists that students notate their equations correctly: 'But remember that you need to use parentheses around each array to make the equation correct.' In addition to this, the teacher also uses another context/ visual model – the curtain arrays – to strengthen the work with arrays. (Action Researcher 5)

Two action researchers observed that teachers used multiple modalities for representing a problem (e.g., visual, verbal, tactile):

During her lesson, the teacher shows that she values various representations for the same problems. She also uses multiple sensory representations, which is verbal, visual and tactile. The students used open arrays, pictures of eggs in rows of 5, they used their fingers to skip count. The teacher drew a visual representation of each of these on the white board. The teacher also presented array cards for the students to use. (Action Researcher 3)

The teacher encourages multiple student representations and facilitates discussion about these representations so that students can express their mathematical ideas. For instance, for the problem presented at the beginning of the class period, the teacher encouraged a student to share her representation—drawing hash marks inside three circles to divide 18 by 3. The teacher then encouraged another student to represent his thinking when he wrote a number sentence on the board. The teacher modeled her own mathematical thinking during the Missing Factors game when she pointed to a dimension on an open array she had drawn on the board and asked out-loud, 'What is this telling me?' The teacher regularly encourages the use of visuals and manipulatives so that students can think about a problem or concept. For instance, the open array encouraged students to make connections between arrays, skip-counting and multiplication. When the teacher wrote on the board $5 \times \underline{\quad} = 35$, she

encouraged students to think about the relationship between factors and dimensions. Students then skip counted by fives to 35 in order to find the missing factor, 7. Another student made the open array a closed array by drawing lines within the array and realized there were 7 columns. Different representations used in this situation encouraged students to find the missing factor using a variety of methods. (Action Researcher 10)

In one classroom, children created posters to showcase their mathematical thinking of “big ideas” as the action researcher wrote:

When creating posters of their work about the candy boxes, their posters clearly explained their thinking after revising their work when others had commented and questioned them. Although I did not have the opportunity to return the next day and see the completed work, during the post interview the teacher said, ‘Every poster had a coherent idea. When students got their posters back after the gallery walk, we took one as a class, looked at the Post-its, narrowed them down to find the best questions. When we were finished two students immediately said they knew how to ‘fix’ their poster.’ All the students identified a big idea and were able to state and represent that clearly. One of the goals for the lesson was ‘letting the work (poster) speak for itself.’ ‘This is so different from the beginning of the year. The posters were focused on proving one specific big idea. And students were able to communicate with each other...It might be a good study to save Post-its throughout the year, to follow students’ development as questioners, analyzers, interpreters.’ (Action Researcher 4)

Use of Contextual Problems. A central rationale for teaching mathematical thinking is that it is necessary in the world beyond the classroom. The world beyond the classroom includes real-life situations that present mathematical problems as well as natural occurrences of mathematics in the world. Most of the classes observed for this report included real-world contexts given to mathematical ideas in class. Real-world contexts in the observed lessons included: stamp problems, food problems (e.g., eggs, candy), and common objects (e.g., markers to share equally).

Action Researcher 9 noted the enthusiasm in her teacher’s use of contextual problems as she recounted part of an interview:

Since we began doing certain units of study, those units are open and allow kids to explore problems with others, to problem-solve with others. The open investigations help promote math thinking...As a process the first step is to have that understanding of the concept you’re exploring, what information the problem is giving you, you need to understand it, you need to make connections between what I know to help me figure out the relationship between what I know and what I am exploring. Once you problem solve and decide on a strategy, you need to take that and apply it in the real world. Math thinking is evident if a child can take the thinking out and use it in the real world. That’s the final stage-when they apply it in the real world...Because the problems are open, students go off and try to find a way to solve it and they’re forced to use what they know and build from there, onto that.

Other Instructional Practices. When coding the classroom observations, several action researchers had data that they believed did not fit into the five indicators instructional practices. Additional categories which they created included: differentiation, contextual tools, big ideas and using relationships, use of counter examples, and review/reiterate.

Differentiation. Two action researchers felt that differentiation warranted its own category. Based on her observations, Action Researcher 1 suggested that “Differentiated Intervention” become a category of teacher instructional practices that pushes kids to think mathematically at their own level of understanding whether it is more or less advanced than the group.

This action researcher noted:

During my observation, I noticed how [the teacher] worked with children to accommodate their level of mathematical thinking. There was one student in particular that finished quickly and said, ‘I’m really good at this.’ [The teacher] told her to try triples and quadruples. After working on this for awhile, she asked if she could use bigger arrays. [The teacher] suggested she start to use ratio tables to keep track of what she was doing, rather than arrays. In this case, a student was ready for more challenges and [the teacher] provided it for her. A good question is whether this child would have continued to use and grow her mathematical thinking if she had not been provided the challenges or would she have become bored and passive?

For a few other children, [the teacher] specifically instructed them to use more sophisticated mathematical thinking. When she saw a child calculating the dimension of an array by counting each square one-by-one, she told the student, ‘We are not counting one-by-one, you need to use a different way.’ For another student, she said ‘You should choose a bigger array, not 4X1 because you can do that doubling in your head. You need to challenge yourself.’ With regard to mathematical thinking, such redirection shows that [the teacher] knows her students as mathematicians. She knows where they are at and where they need to go as mathematical thinkers.

There were 3 students who were struggling with the task. [The teacher] pulled them over to the rug area and worked with them in a small group. She broke down the problem and had them do each step together with her. Again, the question is whether these children would have continued to try to use mathematical thinking if [the teacher] had not intervened with individualized instruction or would they have become bored and passive?

Action Researcher 2 also felt that differentiation warranted its own category as an indicator of an instructional practice to elicit mathematical thinking:

The teacher considers differentiation to be central to providing meaningful mathematics instruction to all students. The math warm-up, because it depended solely on individual students’ reasoning, was open to all students. The day’s lesson, though, was differentiated by difficulty of task, although all tasks were the same—to find the total number of marbles an individual child received after 30

nights. 'You guys have your sheets at your table, with your name on it—that is the child you will figure out. Get right to work. Your names are on top of the sheet. I already chose for you.'

In her questioning as she conferred, the teacher scaffolded or stretched individual students: 'What if I asked you how many marbles she has on the 19th day?' vs. this interchange:

S: That is a lot of 5s.

T: How are you going to keep track:

S: There are so many 5s; they are running all over the place.

T: How are you going to keep these 5s organized? After one night he has 5 marbles, how many will he have the next night?

S: Silence, then $5 + 5 = 10$ so 20, is what $10 + 10$ is,

T: Do you think you could draw a picture to keep track I like how you are using what you know to figure out how to add the 5s together. You are going to draw columns of all the marbles he gets? I am going to check back with you later.

In both cases, she leaves it to the child to carry out the work, but the work they are doing is quite different.

The teacher speaks positively about the work during the math lesson because it was engaging for nearly all the students, despite different levels of understanding: 'And for the most part the kids seemed to be engaged in the work, when I was going around, they were grappling with what to do. You could ask them a question, they had to stop and think about it, and they did have to think, there was always a little something they had to think about it.' The task was successful because 'The majority of the kids made choices that made sense to them when they were working independently.' But also 'For some kids it is very daunting not to have a clue where to start. I think it is good that they get to do this and have a little more structure in the next few days.'

Contextual Tools. Two action researchers noted they had observational data which fell under "context" but not necessarily a contextual problem. In one case the action researcher felt that the teacher's explanation of the purpose of the activity was setting the context for the lesson:

The purpose of this activity today was to get you to see that if you know some smaller facts you can build these together to learn your bigger facts. (Action Researcher 5)

While in another case, the action researcher believed that the games format was a contextual tool as she noted:

The teacher uses games to engage all students and give them a safe environment to discuss their thinking with one another. While not a contextual "problem" in the traditional sense, the games serve as a contextual tool for the students to explore mathematical thinking. In her class, the children played three games: Circles and Stars; Array Card Game; and Divisibility Game. (Action Researcher 3)

Big Ideas and Using Relationships. One action researcher reported that the teacher she observed continually referenced tying her teaching to the big ideas of mathematics and using relationships to understand new things. In discussing what mathematical thinking means to her Teacher 2 connects it to the process of figuring out something, of creating new knowledge. She said:

So using the truths of the number system, using the number system the fact that it is a base 10 number system, help you figure out how to subtract or add using the elements to manipulate the numbers in the number system to use equivalence to help you figure out things...Problematic situations, just having to figure something out, using all these different things that you know to figure out something else, that wonderful feeling when you figure it out when you didn't know before. (Teacher 2)

This teacher explicitly stated that she helps her students think mathematically by connecting them to big ideas:

Having the big ideas from Randall Charles, having that in my mind when I'm planning, and having the ideas up for them to refer to, starting to see some students use that language to express their thinking. (Teacher 2)

Teacher 2 further stated that asking students to generalize beyond a particular problem is important when helping students to think mathematically. The teacher again referenced big ideas when answering the question how she assesses student's thinking while teaching:

I try to bring them to the truth of mathematics, not just this random thing that you saw in this instance, or that someone told you, bring it back to the number system or this pattern. (Teacher 2)

During the math warm up, the teacher asked students to consider the equivalence of two sides of an equation, coming back to that equivalence as the basis of reasoning.

Counter Example. One action researcher felt that a sub-category of reasoning and proof is the counter example. When asked how a teacher assesses mathematical thinking while teaching, the teacher responded:

The use of counter examples, for example, when they are saying it worked and then they reference a method that only happened to work because of specific numbers, I ask them to try it with a different numbers that wouldn't result in the correct answer with their method, a counter example to their [over] generalizations. (Teacher 2)

Review/Reiterate. Action Researcher 8 noted that the teacher she was observing asked students to review what strategies they used to solve the problem before they went off to do the independent activity. She felt this type of discourse warranted a separate category.

Questions for Your Consideration

- *How might network-wide professional development expand teachers' expertise in using questioning systematically to promote students' mathematical thinking?*
- *What effective mechanisms (e.g., grade level meetings, faculty meetings) should schools have in place for teachers to share various classroom practices?*

Outcome: Student Demonstration of Mathematical Thinking

The theory of action of the PAN 3 Formative Assessment/Instruction Cycle in Math posits that teacher participation in each of the four core practices will foster the development of student mathematical thinking. To better understand the outcome of the core practices the following research question was posed:

Research Question 1a): What impact does teacher participation in a formative assessment/instruction cycle in math have on the development of third grade students' mathematical thinking?

To investigate this question, action researchers conducted one classroom observation of the teachers participating in this and their students (See Appendix C for the action researchers' summaries of observed lessons). Action researcher then coded the observation data along the six indicators of student demonstration of mathematical thinking: (1) reasoning and evidence; (2) observation, conjecture and generalization; (3) making connections; (4) communication; (5) strategizing and; (6) mathematical representation.

1. Student Demonstration of Mathematical Thinking

Action researchers observed all six indicators of student demonstration of mathematical thinking across the classrooms in this study with varying degree and frequency. What follows are highlights of the observed student demonstrations of mathematical thinking.

Reasoning and Evidence. Mathematically proficient students make sense of quantities and their relationships... Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades (Common Core Standards, 2010, p. 6).

The expectation for students to reason and give evidence was of great importance to the classroom experiences of the students in the PAN 3 network. The action researchers observed different forms of reasoning including reasoning about patterns, reasoning about parts and wholes, reasoning about the relationship between operations.

When students reason they often use patterns and relationships they see in numbers to explain and justify their thinking. A student in Teacher 7's class explained why she knew that 6×4 is 24 by offering the following model, "Well, I know that 6×5 is 30, so just take away one group of 6, that's 24".

Reasoning about numbers also requires that students think about the relationship of the parts and the whole in numbers. During a multiplication string activity in Teacher 4's students were asked to solve 14×5 . One student explained,

I took the 4 off 14. $10 \times 5 = 50$. $8 + 8 = 16$. 8 is 2 4s. Another 8 is 2 more 4s. $16 + 4 = 20$, plus one more 4. That's the 5 4s I took off. $50 + 20 = 70$.

When students are asked to reason about a problem, debate and disagreement often occurs. In Teacher 5's class the students were shown the images of curtain arrays. The students had an opportunity to articulate their thinking when asked how many objects there were on the curtains. When one child said, "I think it's 14 because I counted 2 down and I think I saw 7 across," another child refuted this because "half of the shade is closed," so half of the objects were obscured, thus "making the total 28, not 14."

In two classes students were asked to reason about the relationship between multiplication and division. In Teacher 10's class students were asked to prove why they thought a problem was division and not multiplication. One student said, "Because you have to split 18 in 3 ways." While in Teacher 3's class there was this discussion of multiplication and division.

T: Why can we see it as times or division?

S1: If you have 25 then it means 5 rows of 5 inside.

S2: When you times something like 5 times 5 or 4 times 5 and you know the answer, it is easier to divide it because you know the answer and it's going to be that other number.

Observation, Conjecture and Generalization. Fostnot and Dolk (2002) use the term "mathematizing" to describe the way students construct meaning. They write,

Children are organizing information into charts and tables, noticing and exploring patterns, putting forth explanations and conjectures and trying to convince one another of their thinking (p. 4).

In many lessons observed there were moments when students made observations and conjectures about what they were thinking about. In Teacher 3's class, one student made this observation.

S: I think division is like subtraction over and over again. So 24 minus 3 is 21 and 21 minus 3 is 18 and then when we get to 0. We count how many times we subtracted.

Meanwhile there were three lessons in which observation, conjecture and generalization were central to the student thinking. In Teacher 1's lesson students were looking at area to understand the effect on the area if you double one of the dimensions.

T: So what about this new array. First it was 3×4 , then Josie drew the new array, which is 3×8 .

S1: The area doubled.

T: Because why?

S2: The 4 doubled.

S1: Because if you double either dimension you have to double the area.

In Teacher 4's lesson students were asked to think about the big ideas in multiplication and make generalizations from their work with 2×5 arrays and explain them on posters. The students came up with following conjectures based on their observations and noticing of patterns:

- *No matter how many times you halve and double the array the answer stays the same.*
- *If you have an odd number of 2 x 5 arrays the box will still be a multiple of 10 because each 2x5 array is 10.*
- *Each time we add an array it's a different multiple of 10.*
- *By using the x10 rule you slide the number in the ones place and put a zero at the end of the number.*
- *All the numbers we make with the 2x5 arrays are multiples of 10.*
- *You can't make odd numbers with a 2 x 5 array.*
- *When you double and halve the array the product stays the same."*
- *When you double one factor, the product doubles.*
- *There is a 2 times fact for every multiple of 10.*

Teacher 2's lesson was different from the other lessons. Students were asked to make conjectures and predictions based on the patterns within a contextual situation.

The lesson was an introduction to a unit on linear function using a story context where children in an imaginary land received a certain number of magic marbles every night for a 30-day month each year. Some children have marbles left over from last year, others don't. Different children get different amounts each night. Whatever number of marbles they get the first night they will continue getting every night for 30 days.

Here was a discussion between the teacher and two students.

T: Bolar had 30 marbles left over from last year. This year he was given 5 on the first night, and then 5 every night for 30 nights. What about you, [Student 1]? What is going on? Why did you start with 30?

S1: Because Bolar had 30 left over from last year.

T: Did you compare your representations [with Student 2]? How did he get 60 on the 10th night?

S1: I guess 10 x 3 is 30 and since she already had 30.

S2: Then I decided to add 30.

T: How come you added 30 every time on your table?

S2 (smiles): I thought it was a faster way,

T: How did you know to add 30?

S2: 10th cube is 30 so you just add for the other 10th cube you add 30 every 10 days you would be adding

T: Here he is going 10 nights to 10 nights.

And Teacher 2 had this discussion with another student to try to get him to generalize an approach.

T: Tell me what you did.

S3: In the beginning she had 62, I know how to count by twos [using] a ratio tale

T: Interesting

T: How many marbles would she have on the 24th day? Would you be able to figure that out? Could you do something similar to see what she would have after 24 days?

Making Connections. Mathematically proficient students look closely to discern a pattern or structure.

Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$ (Common Core Standards, 2010, p. 8).

This ability to see connections between different mathematical representations and ideas helps students to see the structure within mathematics. In the ten classrooms we see various instances of students making connections.

In Teacher 9's class we students began to grapple with the idea of the commutative property. The teacher said, "We would call this a 6x2 box" (she says as she points to the lines and columns) and she points to the 2x6 box. Then she asked the student to name it, drawing their attention to the fact that it is turned. Students drew boxes next to each other because they said the boxes were connected. They also illustrated connections with the use of an arrow. Teacher 9 said in an interview:

Mathematical thinking is more of a process where anyone shows their ability to reason, to problem solve, to make connections between the concepts and ideas they know and build relationships between those concepts.

In Teacher 7's class there was an "aha moment" had by a student in developing a deeper understanding of doubling a dimension of a rectangle. The student made a connection to a strategy that was shared earlier by another student: She said, "He did the same thing as Serafina. She doubled the array lengthwise. He doubled it going the other way." She points out that it is the "same thing" or the same product if you double the length or double the width, or likewise one of the factors in a multiplication combination.

A large underlying idea in making connections is when students are able to use prior knowledge to construct new knowledge. Connections were a central idea in Teacher 5's lesson called *The Big Meal* game. It was designed by the teacher based on the idea that students could use what they already know about smaller multiplication problems to solve harder problems. In the post-observation interview, the teacher acknowledged that her students "already have the tools to solve these harder problems." So, the activity was built upon the idea that they would already be making connections just by finding the smaller arrays they knew to fit into the larger arrays. In this instance Teacher 5 was working with Student 1, a student who struggles with mathematics. She was working on 6×7 . She found a 3×6 "side dish" and is figuring out where to place it on the large "plate."

T: So far you have 3×6 , [Student 1], that's good. Remember you want to try to keep one dimension the same and when you're placing it on the array, keep it arranged from edge to edge.

Teacher placed the 3×6 correctly on the 6×7 array.

*T: Count the squares and figure out what you need to look for."
(Student 1 counts the squares in the grid.)*

S: I need to find a 6 x 4 piece, right?"

T: That's right. So see if you can find a 6 x 4 side dish and make sure it fits on the plate."

(Gaby finds the 6 x 4 array and places it in the remaining space on her "plate.")

T: So how do you write it?"

Student 1 wrote, $6 \times 7 = (3 \times 6) + (4 \times 6)$. Student 1 then wrote '18' below the (3×6) portion of the problem, but seemed to get stuck on the (4×6) portion and you could see her using her fingers to skip count under her breath: 6, 12, 18, 24.

T: So [Student 1], it doesn't matter how we cover the Big Plate; we'll still get the same answer, which is 42.

In Teacher 3's class a student made a connection to money as she divided 52 by 2.

S: Quarters. 52 reminds me of 50 so I think of a number that could be close to 52 like 50 cents. 25×2 is 50 cents. So if you just go up to 26 then $26 \times 2 = 52$.

An interesting occurrence happened in Teacher 7's class. During the partner work time, a student picked 9×5 and said the following:

S: I know 5×10 is 50 so... wait. If 5×10 is 50 I need to minus one.

This student is on the verge of making some important connections, but was getting tripped up in his thinking. However, because he is talking through his thinking and approach it is easy to see exactly where his confusion lies and for the teacher to help get him back on track.

Communication. According to the Common Core Standards,

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately (2010, p. 7).

Classroom communication can take the form of teacher with the whole class, teacher to small groups or individual students, and student to student. All of these forms of communication were observed in each of the classrooms. In Teacher 1's class a dialogue about the meaning of area led students to a more precise definition of the term.

T: As you move to the rug, I want you to think about the word "area"; what does it mean in math?

T: So, what does area mean? (First few students called on have their hands raised, then T calls on a few more students who do not have hands raised).

S: It means like 6×6 .

T: So that's an example of a way to calculate area. What else do you know about area? How else could you describe it?

S: It's the middle of an array.

S: Area is a square.

T: When we find the area, what are we finding the area of?

S: [It is] the area of a space inside.

In Teacher 6's class, two students working independently of an adult, had a conversation about a set of division problems.

S1 draws circles to represent the tables and puts people around the table. She gets 9 as the answer for the second problem

S2: I think all of the answers are 12.

S1: (disagrees and looks over at her partner's work.) You should look over the second problem.

S2: Oh, I need to correct that.

In Teacher 10's class students communicated with each other about whether the problems were multiplication or division.

S1: There are 2 kids and 30 pretzels. How many pretzels should each student get? What plus what equals 30?

S2: $15+15=30$.

S1: Yes, is that multiplication or division?

Together, these two students talked through the problem and the student who was unsure at first grew to understand that the situation was a division situation.

In Teacher 4's class students were asked to construct a big idea as they looked at different arrays formed from a 5 by 2 array. Two students had the following discussion.

S1: What do you think our big idea is?

S2: If we add different numbers of 5×2 arrays it will always be a friendly number like 10, 20, 30, 40, 50 because each 5×2 is 10.

S2: Six 5×2 arrays would be 60.

S1: Two 5×2 would be 20. Three 5×2 would be 30... It's always an even number.

They went to the teacher to share their discovery.

S2: It would always be a multiple of 10.

S1: 16×10 would be 160.

T: What big idea can you convince people of?

S1 & S2: Multiples of 10.

T: Why is that happening?

Strategizing. In a constructivist classroom students are asked to create different strategies to solve problems. In a third grade classroom students are developing multiple strategies to both multiply and divide. "Developing all these strategies is no mean feat! The progression of strategies... is an important inherent characteristic of learning (Fostnot & Dolk, 2002, p. 11). In the mathematics lessons observed by the action researchers students were encouraged to use

multiple strategies as a means of developing their number sense. This was encouraged by all the teachers, often with the goal of finding the most “efficient” strategy.

In Teacher 7’s class the students applied and shared a variety of strategies for solving multiplication problems with arrays. Some of the strategies involve doubling and/or partial products. Here they were looking at the problem 4×6 on an array.

S1: I know $2 \times 6 = 12$, then add another 12.

S2: I did it by 4. I counted by 4 in three columns, that was 12, then I doubled the 12 because there is the same number on the other side – half of 24 is 12.

A third student figuring out 4×9 stated, “if you know that $2 \times 9 = 18$, then add $18 + 18$ to figure out 4×9 .”

A student in Teacher 4’s class showed an understanding of the distributive property as a strategy for multiplying when she solved 14×9 as $(9 \times 4) + (9 \times 10)$.

Teacher 8 used a classroom discourse to have students share their strategizing.

T: Today I ask you to take a look at the set of stamps and tell me how might you figure the value of the stamps.

S1: $5 + 5 = 10$. Another $5 + 5$ is 10. Another $5 + 5 = 10$. Another $5 + 5 = 10$. That’s four tens.”

Teacher 8 writes what the student said and repeated what she said, “She sees $5 + 5 = 10$ and $5 + 5 = 10$. She’s regrouping. Teacher 8 circled the stamps vertically and the student said that she saw it horizontally. Teacher 8 asked if it mattered and she said no.

T: Does another have any other strategies?

S2: I did skip counting.

T: Something like 5, 10, 15, 20... like that?

T: Heaven, what did you do?

S3: I saw 4 fives and know it as $20 + 20$.

Teacher 8 repeated what Heaven said and modeled it on the overhead.

T: When you see it as $20 + 20$, it is like seeing it like 2×20 .

T: (To another student) So when you see the four 10’s, how do you write that?”

S4: Four groups of 10

T: Look at [the first student’s work] $5 + 5 + 5 + 5 + 5 + 5 + 5 + 5$. It is like seeing $(2 \times 5) + (2 \times 5) + (2 \times 5) + (2 \times 5)$. She’s regrouped her five and seeing it as 2 groups of 5 and seeing it as 10. What do you think? Any thoughts? Comments?

Mathematical Representation. When students gain access to mathematical representations and the ideas they represent they have a set of tools the significantly expand their capacity to think mathematically (NCTM, 2000, p, 67).

We can observe the importance of representations in the following classroom discussions.

In Teacher 4's class students were using open arrays to represent multiplication problems. She wrote 14×9 on the board.

S1: 14×10 for a ballpark estimate.

T: Can I draw an open array?

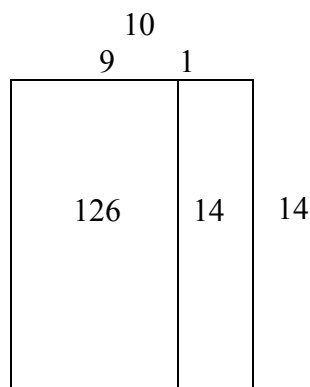
S1: Yes. 14 on the side. 10 on the top. 14 in the middle. $140 - 14 = 126$.

T: How would that look on the open array?

S1: You'd cut off part.

S1: On the side.

T: It would look like this?



Another student represented the problem quite differently

S2: I did $9 \times 4 = 36$ because $(9 \times 2) + (9 \times 2) = 36$.

$9 \times 10 = 90$.

$90 + 36 = 126$.

Teacher 2's students chose to use diagrams and ratio tables to try to figure out how many marbles there would be after 30 days. Student 1, had a ratio table that showed 10/60, 20/90, 30/120.

T: What is [Student 1] doing?

S2 I think he was going every 10 days... He is making it an easier way for him

T: The first night he has 60 including the leftovers, and then he went by 30s

How come he went by 30s?

S2: Because every 10 nights Franick gets 30 new marbles, if she gets 3 marbles each night, after 10 nights she has 30 new marbles.

Questions for Your Consideration

- *What teaching practices will promote students' development of the habits of observing, conjecturing, and generalizing in mathematics?*

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VI. Appendices

Appendix A: Data Collection Instruments

Teacher Interview Protocol

PART I: PRE-OBSERVATION QUESTIONS

1. What does mathematical thinking mean to you? Can you give me examples?
2. What do you do in your class that helps your students to think mathematically? Can you share some examples?
3. How do you assess whether your students are thinking mathematically? [*Probe: how do you assess their mathematical thinking while you are teaching?*]
4. What is your understanding of the purpose of the Math DY0 initiative?
5. In what ways have you participated in the Math DY0 initiative and for how long? [*Probe: assessment design, assessment administration, assessment scoring, Network IIMs, school-based scoring and implications work*]
6. This is a copy of the third Math DY0 assessment for third grade. Can you share with me how the assessment reveals students' mathematical thinking? [*probes: how does student work on the Two-Pen questions reveal student mathematical thinking? How does the student work on the story problems reveal student mathematical thinking?*]
7. How do you use the DY0 math assessments? [*probes: how do you use student data from the Math DYOs in your planning and instruction?*]
8. Has participating in the Math DY0 impacted your everyday teaching in any way? If so, how? Can you give me specific examples? If not, why do you believe it has not impacted your everyday teaching?
9. Has participating in the Math DY0 impacted the development of your students' mathematical thinking? If so, how? Can you give me specific examples? If not, why do you believe it has not impacted the development of students' mathematical thinking?

PART II: POST-OBSERVATION QUESTIONS

1. What were your goals for the lesson?

2. How were you trying to elicit mathematical thinking during this lesson? [*probe: What parts of the lesson/activity were intended to elicit mathematical thinking?*]
3. How do you feel the lesson went? Did you achieve your goals for the lesson? What did you observe in students' behavior that leads you to that conclusion?
4. How do you think your students demonstrated mathematical thinking during this lesson? [*probe: Can you give me specific examples?*]
5. In our previous interview, you said that mathematical thinking was [. . .]. How does your definition of mathematical thinking connect to the examples of students' mathematical thinking that you just described?
6. Earlier, you described participating in professional development or a meeting (in the form of the IIM or having the information from the IIM turnkeyed by another teacher). What did you learn from the IIM or turnkey experience? How did that information influence your planning for today?
7. How did the student data (scores or rubrics) for IA3 inform the way you planned your lesson today? In what way was your thinking about teaching
 - Individual students
 - Small groups of students
 - The whole classdifferent as a result of the information you received from this interim assessment?
8. If you could go back, would you change any decisions that you made in this lesson? [*probes: If yes, what would you change and why? If not, why would you not change anything?*]
9. What did you learn from teaching this lesson? How does this inform what you will do next?
10. Is there anything else we should know about the math DYO process? Your definition of mathematical thinking? Your students' mathematical thinking? The use of math DYO data in your planning process?

Classroom Observation Protocol

PRE-LESSON NOTES

Prior to the lesson, have short conversation with the teacher to find out the topic, goals, and general plan for the lesson. These will help you better follow the action.

School:

Teacher:

Date:

Lesson Topic:

Teacher's Goals for Lesson (from pre-lesson conversation):

Classroom Set-Up (student seating, teacher position, etc):

Materials Used to Start Lesson (on board, posters, handouts, etc):

DURING LESSON NOTES

During the lesson document the following: (1) what the teacher says; (2) what the teacher does; (3) exchanges between the teacher and students; (4) what students say (including to each other); and (5) what students do. Record time at regular intervals.

Time

Activities

POST- LESSON NOTES

What questions do you have for the teacher following this lesson?

What instructional decisions do you want to follow up on in your follow up interview?

What instances of mathematical thinking did you observe that you plan to discuss in the post-observation interview?

What other information do you need in order to understand what went on in this lesson?

Appendix B: Data Coding Instruments

Indicators of Teacher Instructional Practices That Elicit Student Mathematical Thinking

INDICATOR	DESCRIPTION	EXAMPLES
CLASSROOM DISCOURSE	Classroom discourse consists of any of a variety of exchanges in which ideas about mathematics are communicated within a classroom. As an instructional strategy, this may include dialog between teacher and students, dialog among students, presentations, use of images or diagrams, or any of a host of communicative methods to share mathematical ideas.	- Teacher communicates expectations for students to explain their thinking process(es) and defend them to one another. - Teacher takes a neutral stance, encouraging students to discuss and argue about mathematical ideas. - Teacher poses a problem to students, and leaves student to their own facility and collaboration to find a solution to the problem. - Teacher allows for student-driven discourse in groups.
OPEN-ENDED QUESTIONS/PROBLEMS	Asking open-ended questions is a key strategy teachers use to stimulate mathematical thinking. Open-ended questions can embody the demand for student exploration, sense-making, application of ideas, extension of ideas, construction of new ideas, and struggling with the unknown. Teacher provides for multiple entry points that can lead to one solution or multiple solutions.	Teacher poses questions such as: - Can you explain what you are thinking? - How can we use what we know about working with small numbers to help us work with larger numbers? What's the same? What's different? Teacher presents a problem such as: - How many ways can you make 17 cents?
"HOW" AND "WHY" QUESTIONS	Asking "how" and "why" questions gives students an opportunity to describe what they did and why they did it. The students have to be able to give reasons for the choices they made. Reasoning through problems and offering evidence in support of ideas are essential components of mathematical thinking. This leads to the development of student meta-cognition	Teacher requires students to support their answers with evidence. - How do you know if your answer is correct? Teachers ask students to describe how they reasoned through a task. - What did you do? Why did you do it?
MATHEMATICAL REPRESENTATION	Mathematical representation refers to multiple ways in which mathematical concepts are presented. Because one specific idea can often be presented in many different forms – in number, in the English language, in diagram – it is important to the development and expression of students' mathematical thinking for them to understand how they can create mathematical representations to express their mathematical ideas.	Teacher records multiple student representations and facilitates discussion about the connections among these representations. Teacher models her own mathematical thinking through the questions she asks herself. Teacher encourages the use of visuals and manipulatives so that students can think about a problem or concept.
USE OF CONTEXTUAL PROBLEMS	A central rationale for teaching mathematical thinking is that it is necessary in the world beyond the classroom. The world beyond the classroom includes real life situations that present mathematical problems as well as natural occurrences of mathematics in the world.	Teacher presents a problem such as: A group of 6 is given 4 one-foot sandwiches. A group of 8 is given 6 one-foot sandwiches, and a group of 12 is given 8 one-foot sandwiches. If sandwiches are divided equally in each group, in which group will the students get the largest sandwich?

Indicators of Student Demonstration of Mathematical Thinking

INDICATOR	DESCRIPTION	EXAMPLES
REASONING AND EVIDENCE	Reasoning is the capacity for logical thought, explanation, and justification. Students reason through problems and offer evidence in support of ideas. An aspect of reasoning is reflection. Students reflect on their own thinking, looking at whether their approach is sensible or if there is a need for self-correction.	Examples of students demonstrating reasoning and evidence-giving behaviors include explaining their reasoning process, giving numerical proof, and giving visual evidence of a correct answer. Examples of student reflection behaviors include justifying a claim, refuting a claim of a classmate, correcting their own claims, and investigating a new claim proposed by a classmate.
OBSERVING, CONJECTURING, AND GENERALIZING	Thinking mathematically consists of making observations of available information, making conjectures about possible solutions to problems, and generalizing from a particular case to formulate new concepts for future application.	Students observe different patterns within a set of data, look for relationships within the data and attempt to conjecture a possible general idea.
MAKING CONNECTIONS	New mathematical concepts make sense to students as they are contextualized into the other ideas students have already learned. Students make connections around big ideas in mathematics and constructing new knowledge by making connections to prior knowledge.	Students observe and explain coherence between mathematical ideas. Students observe and explain a connection between the task and past experience, past problems, or other subjects. Students use knowledge or other mathematical understandings to develop a new understanding.
STRATEGIZING	Students with strategic competence not only come up with several approaches to a non-routine problem but also choose flexibly among reasoning, guess and check, algebraic, or other methods to suit the demands presented by the problem and the situation in which it was posed.	Students choose a correct and efficient strategy based on the mathematical situation in the task. Students show evidence of applying prior knowledge to the problem-solving situation. Students build new mathematical knowledge by extending prior knowledge.
COMMUNICATION	Students organize and consolidate their mathematical thinking through communicating their mathematical thinking coherently and clearly to peers, teachers, and others; analyze and evaluate the mathematical thinking and strategies of others; use the language of mathematics to express mathematical ideas precisely.	Students communicate through verbal/written accounts using both familiar, everyday language and formal mathematical language.
REPRESENTATION	Students create and use representations to organize, record, and communicate mathematical ideas; select, apply, and translate among mathematical representations to solve problems; and use representations to model and interpret physical, social, and mathematical phenomena.	Students represent mathematical concepts and ideas using diagrams, manipulatives, and mathematical symbols. Students are able to translate from one type of representation to another.

Appendix C: Action Research Lesson Observations & Analyses

The PAN3 action researchers conducted classroom observations and follow-up interviews in order to address the research questions related to teacher practices that elicit student mathematical thinking, and to student demonstration of mathematical thinking. What follows are the written reports submitted by action researchers. Each subsection provides a summary of the lesson observed at a school, a brief analysis of teacher instructional practices observed, and a brief analysis of student mathematical thinking observed. Selections from these reports were used throughout the report to illustrate findings for the research questions.

Teacher 1

Lesson Summary

The observed lesson focused on creating visuals for the doubling of arrays and expressing the doubling of arrays in an equation. The goal was for kids to see and practice how they can solve multiplication problems through the use of doubling multiplication facts. There was a mini-lesson at the rug and then independent work time. At the end of the math period, there was a brief math assessment. [The teacher] reported that the reason she planned this lesson was based on the DY0 Assessment data. The data indicated that her students were not fluent or efficient in their understanding of multiplication. A lot of them were relying on skip counting and doing so inaccurately. It was also inefficient. [The teacher] wanted to teach them a way to use smaller facts to get bigger facts.

Teacher Instructional Practices that Elicit Student Mathematical Thinking

Classroom Discourse. [The teacher] uses classroom discourse to teach and also to assess students' mathematical thinking. She looks for participation, especially questions and mistakes. She always says to the students, "Having the wrong answer is better than having no answer at all." [The teacher] wants all students to be using mathematical thinking at whatever level that may be. She reports, "Sometimes I will call on kids even when they don't raise their hands. I'll give a problem and say, 'Nobody raise your hand, I'm just going to call on somebody. Everyone should be thinking about this problem.'"

Several times throughout the observed lesson, [the teacher] did say "Hands down, everyone should be thinking about this. I will call on someone." This communicated the expectation for students to be thinking mathematically and be ready to share their thinking rather than being passive during classroom discourse. At the beginning of the lesson, [the teacher] posed the question, "What does area mean?" and allowed for multiple responses from several students. She accepted all answers and restated some as necessary, but the perception was that everyone's participation was valid and that she wasn't looking for one right answer.

At one point [the teacher] had a discourse with a student who was demonstrating understanding of the main idea. [The teacher] questioned the student many times in order for the student to say out loud her math thinking. In this instance, [the teacher] was providing the space and scaffolding (through questioning) for a student to "teach" the math idea.

The seating arrangement was such that 5 students sat at one table or grouping of tables. The exceptions were 1 student who has demonstrated in the past, strong math thinking and skills, but who needs to sit alone in order to concentrate and do his best math thinking and two students sitting in a special location together who are highly distractible and distracting, but are able to accomplish work together. Throughout the independent work time, there was constant conversation among the students and it was mostly related to the work at hand. There were not specific instructions to work independently or to work with a partner or group. Students knew that they could talk with each other about the math work because the space was set up to promote it and this is a workshop format that is always used in [the teacher's] math teaching.

Open-Ended Questions/Problems. [The teacher] started the lesson with an open-ended question and allowed time for multiple answers. She did not correct anyone or indicate that any one student had a “correct answer.” During the lesson, a student shared how he found the dimensions of a doubled array. [The teacher] posed the question, “Did anyone come up with different dimensions? Would any other arrays work?” opening up the discussion for multiple answers. The independent work assignment was open-ended in that students could choose from a variety of arrays to use and they could double the arrays in whatever ways they chose as demonstrated during the lesson.

“How” and “Why” Questions. [The teacher] used language throughout the lesson to prompt students to explain their thinking or procedures and provide evidence. These statements included:

“Because why?”

“Can you think out loud?”

“How did you figure that out?”

“Why did you add $14 + 14$?”

“So how can you figure that out?”

Mathematical Representation. [The teacher] used a variety of ways to represent visually the math ideas she was communicating. She used arrays to start the discussion and to demonstrate doubling. She also used two different color markers to trace arrays to help students understand and organize (by differentiating beginning and end of process) their thinking about what was happening to the arrays. Later [the teacher] addressed the class during independent work time and drew attention to the fact that her “new” array was the entire space outlined in red which INCLUDES the old array.

[The teacher] required students to provide dimension labels for the arrays. Further she prompted a student to state and label the dimensions in a consistent order (width first, then length) which was important in demonstrating different ways to double an array. She elicited and labeled using mathematical terms such as width, length, and dimension.

For the independent work, students used graph paper that allows for organized and mathematical organization of student thinking. At one point, [the teacher] needed to remind the students what they had discovered in a previous lesson, that the size of the graph paper squares were not the same as the size of the array squares.

During a conference, a student was confused and [the teacher] talked her through using splitting and regrouping (which the student was able to do). As the student talked, [the teacher] wrote down the written representation of the student's math thinking on her paper. Conferencing with another student, [the teacher] redirected the student who had put the doubling equation (for the new array) above the original equation (for the old array) emphasizing this organization to help the student keep track of what was going on (starting point to ending point).

Use of Contextual Problems. [The teacher] did not make use of contextual problems in this particular lesson. However, during her interview, [the teacher] did talk about using stories to promote mathematical thinking and understanding. "For example, if we are studying multiplication, we tell stories about things that come in groups. Or when we are studying division, we read "A Hundred Hungry Ants."

Other. During my observation, I noticed how [the teacher] worked with children to accommodate their level of mathematical thinking. There was one student in particular, that finished quickly and said, "I'm really good at this." [The teacher] told her to try triples and quadruples. After working on this for awhile, she asked if she could use bigger arrays. [The teacher] suggested she start to use ratio tables to keep track of what she was doing, rather than arrays. In this case, a student was ready for more challenges and [the teacher] provided it for her. A good question is whether this child would have continued to use and grow her mathematical thinking if she had not been provided the challenges or would she have become bored and passive?

For a few other children, [the teacher] specifically instructed them to use more sophisticated mathematical thinking. When she saw a child calculating the dimension of an array by counting each square one-by-one, she told the student, "We are not counting one-by-one, you need to use a different way." For another student, she said "You should choose a bigger array, not 4×1 because you can do that doubling in your head. You need to challenge yourself." With regard to mathematical thinking, such redirection shows that [the teacher] knows her students as mathematicians. She knows where they are at and where they need to go as mathematical thinkers.

There were 3 students who were struggling with the task. [The teacher] pulled them over to the rug area and worked with them in a small group. She broke down the problem and had them do each step together with her. Again, the question is whether these children would have continued to try to use mathematical thinking if [the teacher] had not intervened with individualized instruction or would they have become bored and passive?

Based on my observations, I would suggest that *Differentiated Intervention* is another category of Teacher Instructional Practices that pushes kids to think mathematically at their own level of understanding whether it is more or less advanced than the group.

Student Demonstration of Mathematical Thinking

Reasoning and Evidence. Throughout the lesson, [the teacher] uses questioning to teach the mathematical ideas. When students suggest answers, she often asks why and students explain or provide more information about their methods or thinking. At one point, [the teacher] stops and

asks the class, “Who understands what I’m saying? Who does not understand what I’m saying?” which required students to reflect on their own understanding.

Observing, Conjecturing, and Generalizing. After visuals of doubling arrays were used several times, the students (as prompted by the teacher) started to make predictions about what the area would be for new arrays. Near the end of the whole group time, a student came up to talk through a doubling calculation. As the teacher poses why questions, eventually the student concluded “because if you double a dimension, you have to double the area.” Through this line of questioning, the student explained what she did and also made a generalization about proportions.

Making Connections. As [the teacher] began to double arrays and have students calculate the dimensions and areas, she did not refer to it as doubling. Then a student asked, “Is this doubling?” and [the teacher] asked back, “Is it doubling?” and the student nodded his head. This student was making a connection between what he saw happening with the arrays and the concept of doubling that he had previously learned. He was making a connection between a new idea and prior knowledge. For one advanced student asking to do more, [the teacher] said, “Do you remember how we used ratio charts to keep track...” Before [the teacher] could finish, the student said excitedly, “Yes! Yes!” and began to use ratio charts rather than continuing to draw arrays. She was using a previously learned organizational/tracking strategy in a new context.

Strategizing. During the independent work time, students chose strategies for calculating the dimensions of a new array that came from doubling an existing array. There were several different methods being used across the classroom indicating that students were thinking mathematically to choose a strategy. Some students used skip counting while others used splitting and regrouping. One student started to use counting by 1s until [the teacher] prompted her to try a different way. Some students used a combination of fact knowledge and splitting and regrouping.

Communication. At the beginning of the lesson, several students attempted to explain what the word “area” means in math. Later a student communicated his dimensions in a different order than had been demonstrated and used throughout the lesson. With redirection, he was able to state it differently. Near the end of the whole group time one student makes a generalization and [the teacher] has that student repeat her generalization to the whole class. [The teacher] has students verbally label using mathematical terms of width, length, and area. Throughout the independent work time, students communicated what they were doing in conferences with [the teacher]. Students also communicated with each other during this time. Usually one student would ask about or attempt to correct another student’s strategy.

Representation. Arrays were provided to the students during whole group time and independent work time. In both situations students were required to represent doubling by creating and labeling new arrays and by writing equations. Using different colors for the arrays helped students visualize the doubling process. Students who were proficient in their understanding were pushed to use other types of representation such as ratio tables. Some students also represented the steps they used in making calculations using splitting and doubling.

Other. During the pre-observation interview and during the post-observation interview, [the teacher] said something interesting about students’ thinking mathematically:

When are they NOT thinking mathematically during math lessons and activities?
When they are completely off-task maybe, but otherwise they ARE thinking mathematically at some level. Kids...are at different levels or stages (of mathematical thinking). Like are they thinking concretely and how far are they from moving toward abstraction?

Teacher 2

Lesson Summary

There are two 2nd/3rd grade classes at [name of school]. Math takes place for an hour at the same time every day—the third grade students from both classes go to [the teacher’s] room for math—the second grade students go to [another teacher’s] room.

While waiting for the students to settle, [the teacher] has a problem on the board:

$$250 + 86 =? 200 + 80 + 6 + 50$$

Is this true?

Students are used to the format where they are asked if an equation is true or false—hence the question mark next to the equal sign. [The teacher] facilitates a conversation around this problem. Because there is not agreement before the time allotted has ended, [the teacher] ends this portion of the lesson by saying: We will keep thinking about these different parts of numbers. The warm up took about 10 minutes.

The main lesson was an introduction to a unit on linear function using a story context where children in an imaginary land receive a certain number of magic marbles every night for a 30-day month each year. Some children have marbles left over from last year. Different children get different amounts each night (for no apparent reason). Whatever number of marbles they get the first night they will continue getting every night for 30 days. The introduction took about 10 minutes.

The children were seated in the meeting area for the introduction and ending, and were seated at tables of 4 – 6 for the work time. The independent work was focused on figuring out the number of marbles a particular child had at the end of the special 30 days. The students were assigned a child from the imaginary planet based on [the teacher’s] estimation of how challenging the problems were. During the work time, children talked freely with each other.

The work time lasted 20 minutes.

During the discussion at the end of class, [the teacher] posted 4 children’s work (see attached) for the same child from Romar and led a discussion about the similarities and differences between the methods used by the four students to figure out how many marbles the child from Romar had.

Teacher Instructional Practices that Elicit Student Mathematical Thinking

Classroom Discourse. The teacher puts a great deal of stress on classroom discourse. In her description of how she helps children think mathematically, she says, “when we come back and have shares [they should] not only explain their own thinking, but understand the others. “ In order to assess mathematical thinking she says “While I am teaching I try to do less, try to have the kids as involved as possible, that we are coming to together as a community, a question that someone brought up or a strategy that someone used, mostly through questioning, explain their thinking, prove their thinking, pushing beyond the answer, while one is sharing the others are

pushed to agree, disagree or not sure with their thumbs, turn and talk, like adding on to each others thinking, or disagreeing.”

The lesson held many examples of the use of classroom discourse. In the opening warm-up the teacher said, “I want to hear your reasoning,” and “We have to convince N or she has to convince us.” Before sending them off to work she made clear the expectation that classroom discourse was at the center of mathematics: “we are going to come back and share your tables and diagrams . . . You want people to be able to understand your representation.” At the end of the work time the teacher starts the share with the instructions “Take a look at each other’s work” and the conversation that follows is in reference to the student work, and how other students interpret that work and explain it to the rest of the class. A student asks another student why did you start at 33? And the teacher repeats the question, and draws the attention of the others to the question and response.

In the post observation interview the teacher responds to the question how were you trying to elicit mathematical thinking by referencing the classroom discourse: “I tried to create a situation where they would express their ideas and add on to each other’s ideas and a community understanding of what someone else did.” And defined the way mathematical thinking was demonstrated in relation to classroom discourse: “Definitely in the share, sharing their ideas, and making sense of others work, use what they knew and extended to this situation.” And again, “I think a lot of them did have the a ha moment when they compared their work and saw they got the same number, or not the same.”

Finally, the teacher specifically references the IIM meetings when she says that what children say gives her insight: “when I pull up alongside someone I have gotten better at seeing a child’s thinking in their work and asking questions based on that work and what they say. This is like the work at the Implications Meeting.”

Open-Ended Questions/Problems. The teacher considers open-ended questions and problems to be central to her mathematics instruction. In the interview she said that she assesses mathematical thinking using open-ended questions: “I go around and ask them to explain what they are working on, and how they are figuring out what they are doing. The task the children were engaged with during the lesson observation also had multiple entry points—the work was to discover a way to figure out how many marbles a mythical child would have after 30 days. During the warm-up the teacher asked many open-ended questions: “E, what do you think?” and “I want to hear your reasoning, why do you think it is true or false?” and an invitation to share thinking: “Tell me more.” As the teacher circulated, she asked many open-ended questions. Some were to help students get started: “What do you think you can do to show your work?” “How are you going to show me that?” Others asked students to reflect on what they were already doing: “How are you going to know when you have enough 2s?” and “Do you think you can draw a picture to keep track?” At other times, the open-ended questions were an extension: “How many marbles would she have on the 24th day. Would you be able to figure that out?” and “What if I asked you how many marbles she has on the 19th day?”

Reasoning and Proof (instead of how and why). The teacher continually asks students to explain how and why they did something, and to connect what they did to the logic of mathematics. In the math warm up she says, “I want to hear your reasoning, why you think it is true or false.” She tells the class, “We have to convince N or she has to convince us.” She asks,

“Why did he say 3×30 is 90 and then $90 + 30 = 120$. Why was this the equation?” In the post interview she specifically references liking the share at the end because the question about the meaning of the numbers created a discussion where students were “trying to understand someone else’s work and add on to each other’s thinking...”

Mathematical Representation. The teacher uses multiple mathematical representations in her work and considers it central to mathematics instruction. She references different models and manipulatives as a support for struggling mathematicians in the pre-interview. In the warm-up she carefully marked up the equation on the board to represent a child’s thinking and to show equivalence.

When sending the children off to work on the day’s lesson, she did not model a particular representation but rather stated “you can use a table, a graph, a diagram but also cautions “don’t put too many decorations or drawings on them, except your diagram or picture or table you need to show how many marbles each child will have after 30 days. You want people to be able to understand your representation.” Students are expected to make sense out of the representations of others, and to make their own representations clear. As she walked around conferring with students, the teacher asked two students, “Did you compare your representations? Can you see what he did in your work?”

The share at the end was a comparison of mathematical representations. “Make sure you can see everyone’s representations,” she said

In the post-interview the teacher explicitly referenced mathematical representations as evidence of the success (or lack of success) of the lesson: “From my point of view I was hoping that kids were clear, how would they label the ratio table, some did, some didn’t what types of representations, I thought some kids would do diagrams or pictures, but I wanted I was hoping to see some way of keeping track of how many nights had passed no matter what system they were using, “ and later “I was trying to have the students make sense of each other’s work figure out what the other person did based on the representation on the sheet.” Later she muses, “And I should perhaps have given them larger paper to work with; it would have been easier to see.”

Use of Contextual Problems. The observed lesson used an elaborate context to support the students in discovering patterns and functions. The teacher took care in setting up the context (“My goals were for them to first of all understand this context that we are going to work with for awhile, I wanted the context to be exciting and make sense “) and felt that the context brought children in immediately to working on the mathematics, no matter their level. She responded to the interviewer’s question about how the DYO results informed her whole class teaching in this lesson by saying: “In this case the whole class teaching was not so influenced, it was a more exciting what can we figure out about this new thing.”

Other Important Math Ideas

Big ideas and Using Relationships. The teacher continually referenced tying her teaching to the big ideas of mathematics and using relationships to understand new things. In discussing what mathematical thinking means to her she said, “So using the truths of the number system, using the number system the fact that it is a base 10 number system, help you figure out how to subtract or add using the elements to manipulative the numbers in the number system to use

equivalence to help you figure out things.” She said “Problematic situations, just having to figure something out, using all these different things that you know to figure out something else, that wonderful feeling when you figure it out when you didn’t know before.”

She explicitly states that she helps her students think mathematically by “Having the big ideas from Randall Charles, having that in my mind when I’m planning, and having the ideas up for them to refer to, starting to see some students use that language to express their thinking.”

She states that asking students to generalize beyond a particular problem is important when helping students to think mathematically.

The teacher again references big ideas when answering the question how she assesses student’s thinking while teaching: “I try to bring them to the truth of mathematics, not just this random thing that you saw in this instance, or that someone told you, bring it back to the number system or this pattern”

During the math warm up the teacher asked students to consider the equivalence of two sides of an equation, coming back to that equivalence as the basis of reasoning.

Counter Example. A sub category of reasoning and proof is the counter example which the teacher used to describe her practice when answering the question how do you assess mathematical thinking while you are teaching. The teacher said: “The use of counter examples, for example, when they are saying it worked and then they reference a method that only happened to work because of specific numbers, I ask them to try it with a different numbers that wouldn’t result in the correct answer with their method, a counter example to their [over] generalizations.”

Differentiation. The teacher considers differentiation to be central to providing meaningful mathematics instruction to all students. The math warm-up, because it depended solely on individual students’ reasoning, was open to all students. The day’s lesson, though, was differentiated by difficulty of task, although all tasks were the same—to find the total number of marbles an individual child received after 30 nights. “You guys have your sheets at your table, with your name on it—that is the child you will figure out. Get right to work. Your names are on top of the sheet. I already chose for you.”

In her questioning as she conferred, the teacher scaffolded or stretched individual students: “What if I asked you how many marbles she has on the 19th day?” vs. this interchange:

S: That is a lot of 5s.

T: How are you going to keep track:

S: there are so many 5s; they are running all over the place.

T: How are you going to keep these 5s organized? After one night he has 5 marbles, how many will he have the next night?

S: Silence, then $5 + 5 = 10$ so 20, is what $10 + 10$ is,

T: do you think you could draw a picture to keep track I like how you are using what you know to figure out how to add the 5s together. You are going to draw columns of all the marbles he gets? I am going to check back with you later.”

In both cases, she leaves it to the child to carry out the work, but the work they are doing is quite different.

The teacher speaks positively about the work during the math lesson because it was engaging for nearly all the students, despite different levels of understanding: “And for the most part the kids seemed to be engaged in the work, when I was going around, they were grappling with what to do. You could ask them a question, they had to stop and think about it, and they did have to think, there was always a little something they had to think about it.” The task was successful because “The majority of the kids made choices that made sense to them when they were working independently.” But also “For some kids it is very daunting not to have a clue where to start. I think it is good that they get to do this and have a little more structure in the next few days”

In the post-lesson interview the teacher says about the impact of the IIM: “Through that and other assessments I anticipated which students would have a more challenging time with this, so I gave them each a problem that would be at a right level of them. I would anticipated some children would veer toward drawing, and I thought of having cubes, I tried to check in with them in the beginning of the things, and they seemed to have a good idea and they didn’t ask and so I thought I would see what they did. The more advanced students, through questioning and sort of when I checked with in them there was a way to check in with them to ask a question to extend their thinking. I thought it would be easier for them but it wasn’t for all of them, they had to do the whole pattern or made a generalization or a small calculation error so there was till enough there”

Student Demonstration of Mathematical Thinking

Reasoning and Evidence. In answering questions in the whole group during the math warm-up, students were continually asked to justify their responses: One student said, to justify her answer that two sides of an equation were equivalent, that “That 2 and that 2, they are both 200, The 200 there is the equivalent to that 200 over there, and the 50 and the 50 over there and there is 80 over here and 6 over here and 80 over there, and 6 over there,” while pointing to the equation on the board.

During the math lesson students continually justified parts of their solution: “Because Bolar had 30 left over from last year. I’m working on Bolar.” And during the share: “Because Franick already had 30. Every night he got 3 more so it would be 3×30 .” “The first night she had she had 30 already. Last year she saved 30 marbles, she saved 30 33, after one night, then 36, then 39 and the next night he has 42.”

Observing, Conjecturing, and Generalizing. Students made observations and conjectures while working on the problems: “10th cube is 30 so you just add for the other 10th cube you add 30 every 10 days you would be adding Here he is going 10 nights to 10 nights.’ And during the share, more observations and conjectures: “I was thinking 30 nights because it makes sense when you do multiplication, because it is that 30 nights and she got 3 marble each night.” Another student: “I think he was going every 10 days. Night 10, night 20, but he was still counting by 3s, doesn’t count each one, he says in 10 s: nights, 30. He is making it an easier way for him. The first night he has 60 including the leftovers, and then he went by 30s.

Making Connections. Many students made connections to previous work. Some connections were to addition or multiplication combinations that the students knew. Other connections were to strategies for solving a problem. For example, one student said, “In the beginning she had 62,

I know how to count by twos, using a ratio table.” Another student held on to something she knew in the face of a, for her, daunting task:

S: That is a lot of 5s.

T: How are you going to keep track:

S: there are so many 5s; they are running all over the place.

T: How are you going to keep these 5s organized? After one night he has 5 marbles, how many will he have the next night?

S: Silence, then $5 + 5 = 10$ so 20, is what $10 + 10$ is.”

In the math warm-up, a student made a connection to the big idea of equivalence when justifying her response: “The 200 there is the equivalent to that 200 over there, and the 50 and the 50 over there and there is 80 over here and 6 over here and 80 over there, and 6 over there.”

Strategizing. Student papers give a lot of evidence of strategizing. Some students were explicit about why they chose a method: “I thought it was a faster way.” [When discussing using multiplication to solve the problem, although that was not what the teacher meant by her why did you multiply? Question]

Communication. Students communicated their thinking both in solving problems and during the warm-up and share at the end. Some communication is evident from the student work. Other communication took place within the context of the work time. Mathematical vocabulary used in the course of this lesson included ratio tables, equivalence, doubling, multiplying, each, and every.

The Warm-up at the beginning of the lesson, where students were expected to convince others of their thinking, and the share at the end, where students were expected to make connections across different representations of the same problem, resulted in students articulating their thinking: For example, in response to the teacher’s question, Okay N, what do you think? N said: You got the 6 mixed up. It is 280, not 206. This equation could be true.”

And then when the teacher said,

Let us see what others think, another student said: You didn’t take away anything you put them in a different order, and a third student builds on that idea: C points to the 50 and the 6 and says, “It is all the same numbers,”

Representation. See student work for evidence of representations as tool for communication and organization. Ratio tables, lists, columns, diagrams and number sentences were all used in this lesson.

Teacher 3

Lesson Summary

The lesson focused on students making connections between multiplication and division. The teacher was hoping that the students would see that there is a connection between multiplication and division and that they can use multiplication to figure out division problems. She began by posing a division story problem and allowing the community to come up with ways to solve it. Following a discussion about the problem and the strategies students used to solve the problem, the students were given games to play and discuss with their groups. Each group of students was given a game that met their needs for working with multiplication. The first game assisted those children who were still counting all or just beginning to skip count as a means of solving multiplication problems. The second game used arrays to give students a visual model of the connection between multiplication and division as they chose cards and either give the answer to the multiplication associated with the array or given the answer, state the dimensions of the array. The third game was a card game in which the students create large numbers and decide what numbers the larger number would be divisible by. After about thirty minutes, the groups returned to the carpet area to discuss what they had done and what they had learned about multiplication and division.

Teacher Instructional Practices that Elicit Student Mathematical Thinking

Classroom Discourse. The teacher places great value on classroom discourse in teaching mathematics. She expects her students to explain their thinking and justify their answers. She states in the interview, "... the questioning that I ask them when they're talking to me about solving a problem...I really push them to tell me how they arrived at an answer; to talk through what they're thinking." She sets up her students in small groups to play games and this creates an environment where students are free to ask each other questions and acquire information from one another. During the mini-lesson, students spoke directly to other students, and during the games, the students were justifying their answers to one another. Another way the teacher fosters discourse is by changing the focus from "the answer to the problem" toward the process. During her lesson she tells her students, "I think you already know how many groups of 5 are in 25;" and in her interview she states, "During the mini-lesson, with the questioning I was really asking them to show how they knew the answer to something. I put a relatively easy question up on the board for them at the beginning because I knew they would rush to an answer. I really wanted them to think through how they got to that answer." Children are also comfortable enough to make mistakes and to disagree with one another. At one point during the lesson, one child says that half of 76 is 37 and the other child very gently responds 2 times 38 is 76.

Open-Ended Questions. The teacher believes that asking open-ended questions helps students to formulate ideas. She pushes her students to be cognitive of the steps they are using to come to an answer or solve a problem. In her interview when I asked her how she helps her students think mathematically she responded, "I really push them to tell me how they arrived at an answer; to talk through what they're thinking." Some of the questions she asked during her lesson were: "What else do you know?" "Did anyone do it another way?" and as she responded to a student about 25 divided by 5 she said, "Talk a little bit about 25 divided by 5. Why did that pop into your head?" She also asked, "Why can we see it as times or division?" "How do you know that?"

“How” and “Why” Questions. The how and why questions she asked were listed above as: “Why can we see it as times or division?” “How do you know that?” She also asked, “How did it help you?” These questions require the student to answer with more than just a yes or no or numerical value.

Mathematical Representation. During her lesson, the teacher shows that she values various representations for the same problems. She also uses multiple sensory representations, that is verbal, visual and tactile. The students used open arrays, pictures of eggs in rows of 5, they used their fingers to skip count. The teacher drew a visual representation of each of these on the white board. The teacher also presented array cards for the students to use.

Use of Contextual Problems. In her lesson, the teacher used a problem related to the season of the year and an item that children would be familiar with, namely, eggs.

Other. The teacher uses games to engage all students and give them a safe environment to discuss their thinking with one another. While not a contextual “problem” in the traditional sense, the games serve as a contextual tool for the students to explore mathematical thinking. In her class, the children played three games: Circles and Stars; Array Card Game; and Divisibility Game.

Student Demonstration of Mathematical Thinking

Reasoning and Evidence. In the lesson, students were required to explain how they came to an answer. While discussing multiplication and division, the teacher asks, “Why can we see it as times or division?” One student explains that if you have 25 then it means 5 rows of 5 inside. Another student says, “When you times something like 5 times 5 or 4 times 5 and you know the answer, it is easier to divide it because you know the answer and it’s going to be that other number.”

Observing, Conjecturing, and Generalizing. As the group discusses the posed problem, some of the students think that multiplication is the correct operation to use and some think it is division and the teacher asks, “Who is right?” One student conjectures that both are correct and that division or multiplication can be used to solve the egg problem because $5x = 25$ and $25/5 = ?$ will result in a correct answer of 5. During the same conversation, one student conjectures and generalizes that division is like subtraction over and over again. She observes that beginning with 24 she can take away groups of 3 over and over until she reaches 0. Later during the lesson, a student makes a generalization about multiplication, namely, the commutative property when she explains that, “if we do a turn around, it will look the same way.”

Making Connections. As two students discuss the number 87 and what it may be divisible by, they know it is divisible by 1 and then they are considering whether it is divisible by three. One of the students makes the connection between division and multiplication and then further to addition of a repeated amount. She says, “I found a way. 87 divided by 3 is 29.” And she writes, “ 3×29 is $29 + 29 + 29 = 87$.” Another student performing a similar task, makes a connection to money and uses fifty cents to figure out what 52 divided by 2 is equal to.

Strategizing. As the students discuss the problem presented in the mini-lesson, one child begins to use the strategy of skip counting by fives to see how many fives are in twenty-five. She keeps track of the fives on her fingers as she skip counts. Another child figures out the division by thinking about the related multiplication. Yet another uses the idea of an open array. Later in the student activity, students were using the strategy of using a known fact to figure out an unknown fact. For example, when trying to figure out six times four, she explained to her partner that she knew it was twenty-four because she thought of six times three which she knew was eighteen and then she just added one more six.

Building on Prior Knowledge to Construct New Knowledge. Students use what they know to figure out more difficult multiplication. They skip count, use known facts, and use what they know about money such as fifty being fifty cents and quarters.

Communication. The students in this class used both everyday language and mathematical language to communicate their ideas during this lesson. Communication was both verbal and written. During the mini-lesson, for example, the students talked about comparing numbers. They discussed counting by fives and using their fingers to keep track of the counts of five. They discussed visualizing arrays and open arrays to help them figure out how many fives there were in twenty-five.

During the activity, one child communicated to his group about how some of the facts were known to him so he didn't need to draw every star to figure out the total number of stars. He communicated his thinking in picture (written) and words.

Following the lesson and activity, the students were communicating their work to one another. At one point, the students discuss what the word "divisible" means. One student communicates that divisible means, "what something is divided by, like two times what is fifty-two."

Representation. During this lesson, the students were using arrays and open arrays to represent multiplication and division. They were also representing multiplication as skip counting and repeated addition. In one group, they represented their thinking about multiplication as stars inside of circles. The circles were representative of the groups and the stars represented the repeated amount. Some students also represented their thinking about numbers near fifty as money, using two quarters to represent fifty cents.

Teacher 4

Lesson Summary

The lesson began with a multiplication string of related problems with children gathered together in the meeting area to share strategies, defend their thinking, ask questions, and evaluate other students' ideas. The teacher questioned and modeled student thinking, particularly using the open array model. Work with the open array model continued as students (in pairs) continued their poster work from the previous day. The day before, the teacher had posed this problem: a candy maker packs his candies in boxes of 10 arranged in a 2×5 array; what would happen if he wanted to pack larger boxes, using only 2×5 arrays to create the new boxes? Today the focus was on "communication, being clear and concise, convincing others of your ideas, identifying a big idea, letting the work [poster] speak for itself" as students postered their findings about the new boxes. The teacher circulated while students were working, talking with students and questioning their ideas as they postered. The posters were displayed and analyzed by students during a "gallery walk" the next day.

Teacher Instructional Practices that Elicit Student Mathematical Thinking

Classroom Discourse. In the interview the teacher noted that she's been working on "lots more reasoning and proof this year. And communication in math. Not just communication, but convincing others of your ideas...being clear and specific when writing in mathematics...I ask them to convince us. I ask them to restate what another has said. I ask them to show and explain their thinking." She began the lesson asking students, "What is your job right now?" A student answered, "Listen to each other; use good body language; we all need to participate." The teacher added that it was "not just listening to me but to classmates." During the lesson she posed questions: "How did you work it out; can I draw an open array; how would that look on an open array; how do you know; what big idea can you convince people of; why is that happening?" These questions encouraged and promoted student thinking and gave students something to talk about. Children continually explained their strategies as she modeled their work, often on open arrays. For example, when solving 9×14 , the teacher did not do the talking; she did the questioning and modeling. One student described this array: "14 on the side. 10 on the top. 140 in the middle. $140 - 14 = 126$. Another offered $9 \times 4 = 36$ because $(9 \times 2) + (9 \times 2) = 36$. $9 \times 10 = 90$. $90 + 36 = 126$." The teacher said, "Let's see if WE [emphasis added] can draw that on an open array." In the post interview, she noted, "The students were focused, listened to each other, and asked appropriate questions. They were willing to allow others to explain their thinking." She facilitated communication between partners as they postered their work, asking 2 students, "Why is that happening?" leaving the pair to communicate their thinking on a poster for others in the class to read. She allowed students an opportunity to discuss and refine their thinking on their own.

Aside from questioning, she asked students to restate another student's thought. "Can any one tell us what S2 did?" She allowed think time before asking again, "Can someone say it again?" And then provided time to "turn and talk" about the problem in smaller groups before sharing thoughts with the class as a whole. When the children were solving the "box problem" she assigned partners, "pairing students at similar places along the multiplication rubric." Both students in the partnership that I closely observed were actively engaged in mathematical

discussion: “What do you think our big idea is?” “We noticed 5×2 , 5×4 , $5 \times 8 \dots$ ” “Let’s start again.” “ 2×5 , 4×10 if we double.” “Doubling doubles the answer. It would be 4×5 .” “If we add the arrays together we’d get double the answer?” “Yes.”

Although I did not observe the gallery walk, on the next day she gave students the opportunity to question and comment on other students’ poster work. When we met for a post observation interview she said, “Let’s look at their poster. It says: ‘We think that the big idea is; you can only have even numbers because you can’t make odd numbers with a 2×5 array because 2 only fits into even numbers.’ The question on the Post-it says, ‘How do you know that 2 only fits into even numbers?’ Although they haven’t done it yet they are ready to revise, make themselves more clear, convince others of their big idea.” The teacher reflected “the decision I made for the second part of the lesson, the part you didn’t see, allowed students time to think. Limiting the number of posters that students comment on let students pose thoughtful, helpful questions without having to think about putting a Post-it on every poster in the room.”

Open-Ended Questions. The teacher posed many open-ended questions. “What do you think class?” after an incorrect solution to 12×19 was offered by a student. She allowed students time to evaluate the solution, talk to their neighbor, and then share their ideas about 12×19 . Another student offered “I know 14×9 .” The teacher asked, “How will that help us with 12×19 ?” Another student offered 128 as a solution. And the teacher asked “How do you know?” After sharing his strategy, another student concluded “228.” And the teacher asked, “Can someone say it again?” These questions never said right or wrong. They facilitated evaluation, thinking, and more discussion. They allowed all students to participate in the work. And they allowed for more than one way to solve a problem. One student used the partial product method: $(10 \times 12) + (10 \times 9)$. Another used the previous problem (14×19) to solve the problem. $(14 \times 19) - (2 \times 19)$. The class was able to understand both and “Agreed.”

After the string when students went off to work in partnerships with the question what would happen if the candy packer used his 2×5 candy box to create other boxes? During the partner work, the teacher stayed long enough with a pair of students to leave them with a question that might sharpen the pair’s thinking and allow them to refine their work. She left one partnership pondering, “Why is that happening?” when thinking about multiples of 10. She did not assign topics for posters. She allowed students to draw their own conclusions. “What big idea can YOU [emphasis added] convince people of?” After the lesson she concluded, “I didn’t have to do as much questioning and guiding as I thought I would... Every poster had a coherent idea.” Some of the ideas included: “No matter how many times you halve and double the array the answer stays the same; when you double one factor, the product doubles; there is a 2 times fact for every multiple of 10; if you have an odd number of 2×5 arrays the box will still be a multiple of 10 because each 2×5 array is 10.”

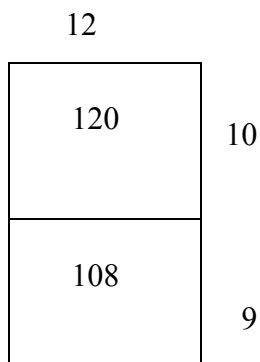
During my time in the classroom I did not hear the teacher or students ask “What’s the answer?” Although an answer or conclusion was arrived at, the work was about thinking, process, evaluation, strategy, modeling, communication. Even when specific multiplication problems were posed they were written on the board without equal signs, leaving them open to discussion, not immediate answers. When one student instantly answered “38” for 12×19 the question “What do you think class?” prompted others to share their strategies and not dwell on the incorrect 38. The open-ended questions allowed every student an entry into the problems and

promoted a range of big ideas from “All the numbers we make with the 2×5 arrays are multiples of 10” to “When you double one factor, the product doubles.”

“How and Why” Questions. The teacher posed many how and why questions throughout the lesson. “How would that look on an open array?” made students describe their multiplication strategies completely and thoroughly in order for the array to correctly represent their thinking. It allowed them to refine their own thinking. When a student offered, “128” for 12×19 , the teacher asked, “How do you know?” The student offered “ $12 \times 10 = 120$.” The teacher drew an open array for 12×10 . The student next offered “ $9 \times 10 = 108$.” After looking at his own array, he concluded “ $120 + 108 = 228$.”

“How will that help us with 12×19 ?” led to using something students knew to help with another problem. Students had previously solved 12×14 and were able to conclude “she doesn’t need 14, she needs 12.” “She’s cutting off the bottom [from the 12×14 array].” The teacher asks “2 what?” One student says to another, “2 19s.” Another concludes, “2 19s. It’s not 14 19s. It’s 12 19s.” And another, “2 19s is 38.” $266 - 38 = 228$. And the multiplication strategy of using one problem to solve another and adjust by removing has been introduced through reasoning and evidence. “We’ve not talked about going past the number and taking away before. Good thinking today.”

Mathematical Representation. In this lesson the open array model for representing multiplication was evident from beginning to end. For example, to solve 12×19 , a 10×12 array was extended by a 10×9 array.



Although this represented the partial product algorithm for multiplication all this work was done mentally without pencil and paper. They knew 10×12 and 10×9 because they used something they knew the $\times 10$ rule (“You slide the number in the ones place and put a zero at the end of the number”) and could add the 2 products together.

When creating posters of their work about the candy boxes, their posters clearly explained their thinking after revising their work when others had commented and questioned them. Although I did not have the opportunity to return the next day and see the completed work, during the post interview the teacher said, “Every poster had a coherent idea. When students got their posters back after the gallery walk, we took one as a class, looked at the Post-its, narrowed them down to find the best questions. When we were finished two students immediately said they knew how to ‘fix’ their poster.” All the students identified a big idea and were able to state and represent that clearly. One of the goals for the lesson was “letting the work (poster) speak for itself.” “This is

so different from the beginning of the year. The posters were focused on proving one specific big idea. And students were able to communicate with each other...It might be a good study to save Post-its throughout the year, to follow students' development as questioners, analyzers, interpreters."

Use of Contextual Problems. This lesson is part of a larger unit about multiplication and arrays. The context for the entire unit focuses around Mr. Muffles' candy store where candies are sold in boxes of 10, arranged in a 2×5 box. Today they were exploring this question: if his candy packer wanted to make larger boxes by combining 2×5 boxes (one layer boxes) what could the dimensions of the new boxes be? Students were able to visualize the boxes (arrays), draw the boxes (arrays), and draw conclusions about how the boxes might change when the number of 2×5 arrays changed. Some of their conclusions included:

"No matter how many times you halve and double an array the answer stays the same."

"If you have an odd number of 2×5 arrays the box will still be a multiple of 10 because each 2×5 array is 10."

"Each time we add an array it's a different multiple of 10."

"All the numbers we make with the 2×5 arrays are multiples of 10."

"You can not make add numbers of with 2×5 arrays."

"When you double and halve the array the product stays the same."

"When you double one factor, the product doubles."

"There is a 2 times fact for every multiple of 10."

Conclusion. The teacher made choices of focus, problems, questions, and context to promote mathematical thinking in the classroom. Mathematical thinking was evident from beginning to end. "We've been working on proof with [our staff developer]. As the students explained their thinking we focused on what makes a convincing argument. I've asked students to focus on one big idea and explain thoroughly." After the lesson, the teacher said, "I saw a range of well-stated big ideas. We will continue to practice expressing our ideas, convincing others of our thinking, and questioning each other. They had knowledge of the content." The DYO interim assessments support this work. The teacher says, "The DYO is a helpful, useful tool, revealing strategies, not just answers...The DYO assessment encourages me to keep doing what we are doing, looking at what kids do and think and making decisions based on what they do, not what the answer is."

Student Demonstration of Mathematical Thinking

Reasoning and Evidence. During the multiplication string students displayed many examples of reasoning and evidence. For example when solving 14×5 , one student explained, "I took the 4 off 14. $10 \times 5 = 50$. $8 + 8 = 16$. 8 is 2 4s. Another 8 is 2 more 4s. $16 + 4 = 20$. Plus one more 4. That's the 5 4s I took off. $50 + 20 = 70$." Another student explained her thinking for $14 - 9$ when the teacher "Can I draw an open array for 14×9 ?" "Yes, 14 on the side. 10 on the top. 140

in the middle. $140 - 14 = 126$.” The teacher “cut off” (removed) 1×14 from the bottom of a 10×14 open array. Another student shared this: “I did $9 \times 4 = 36$ because $(9 \times 2) + (9 \times 2) = 36$. $9 \times 10 = 90$. $90 + 36 = 126$.” Students were able to reflect on their own reasoning: “Oh no, I thought it said something else. 64.” when asked, “How did you get 56?” And students reflected on others’ work: “I liked what she did. She already knew 14×9 . She ‘timesed’ 126×10 ,” but when the teacher asked, “Did she?” another student concluded “ 14×10 ” to be added to 14×9 in order to solve 14×19 .

When creating posters for their work of “making candy boxes” one student explained her thinking to her partner: “If we add different numbers of 5×2 arrays it will always be a friendly number like 10, 20, 30, 40, 50 because each 5×2 array is 10. 6 5×2 arrays would be 60. 2 5×2 arrays would be 20, 3 5×2 arrays would be 30, etc. She then drew 6 5×2 arrays and then drew a 10×6 array. After hearing her explanation and looking at her evidence (her array models) her partner said, “I agree.” After reading other students’ questions and comments about the work on their poster, 2 students immediately said they knew how to “fix their poster.” They were ready to revise, make themselves more clear, and convince others of their big idea.

Observing, Conjecturing, and Generalizing. As mentioned before in this report, students were able to make generalizations from their work with 2×5 arrays and explain them on posters:

“No matter how many times you halve and double the array the answer stays the same.”

“If you have an odd number of 2×5 arrays the box will still be a multiple of 10 because each 2×5 array is 10.”

“Each time we add an array it’s a different multiple of 10.”

“By using the $\times 10$ rule you slide the number in the ones place and put a zero at the end of the number.”

“All the numbers we make with the 2×5 arrays are multiples of 10.”

“You can’t make odd numbers with a 2×5 array.”

“When you double and halve the array the product stays the same.”

“When you double one factor, the product doubles.”

“There is a 2 times fact for every multiple of 10.”

While drawing the arrays (candy boxes), a pair of students had this conversation:

S2: 2×5 , 4×10 if we double.

S1: Doubling doubles the answer?

S2: It would be 4×5 .

S1: If we add the arrays [2×5 and 2×5] together we’d get double the answer?

S2: Yes.

Making Connections. Multiplication strings provide an opportunity for students to make connections between problems when computing. When solving 14×5 , one student said, “I know $5 \times 12 = 60$ add a 5....add another 5.” When solving 14×5 , another student offered, “ $14 \times 5 = 7 \times 10 = 70$. Halve and double.” For the same problem another student said, “I know 14×10 . I halved it.” When the teacher posed 14×19 after 14×9 as part of the string of related multiplication problems, a student said, “266. $14 \times 9 = 126$. Then I did 10×14 . $140 + 126 = 266$.”

While “making candy boxes” 2 students concluded that adding 2×5 arrays would always produce a multiple of 10, and one of the two also concluded that “It’s always an even number,” too.

Strategizing. There are many examples of strategizing shown previously in this report. The class demonstrates flexibility with strategies. The teacher looks “for examples of multiplicative thinking, use of the distributive property, the commutative property, diagrams,” etc. When solving 14×9 , the teacher modeled both of these student strategies for solving the problem:

$$(14 \times 10) - (1 \times 14) = 14 \times 9$$

and

$$(9 \times 4) + (9 \times 10) = 14 \times 9.$$

Communication. The class worked together to make sense of a student’s strategy for 12×19 using a 14×19 array:

S4: “228. Cut off the bottom.”

S5: “She doesn’t need 14, she needs 12. She’s cutting off the bottom. We’re cutting off 2.”

S6: “2 19s.”

S7: “2 19s. It’s not 14 19s. It’s 12 19s.”

S8: “2 19s is 38.”

Class: “Agreed.”

They investigated possible dimensions for “boxes of candy” that could be made from combinations of 2×5 arrays. While creating the posters of their thinking, the teacher said, “They were able to narrow down and revise. They were able to deepen their work. They were able to talk thoroughly about one idea...They used mathematical language. They were specific. They are now focusing on the mathematics, taking time to understand someone else’s work, not just their own.” She said she “holds students accountable for their language. [One student is now] better able to be specific.” For example, the student asked another, “Why did you choose the number 100?” instead of just saying she did not get it.

As partners got ready to poster their big idea, they had this conversation:

S1: “It’s always an even number.”

T: “So it’s always a multiple of 10?”

S1: “Yes.”

S2: “It would always be a multiple of 10.”

S1: “ 16×10 would be 160.”

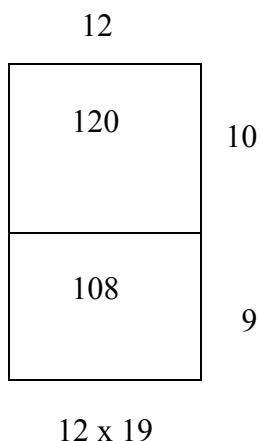
T: “What big idea can you convince people of?”

S1 and S2: “Multiples of 10.”

They used appropriate mathematical language and vocabulary: multiples, even numbers. Others used the words “product,” “factor,” “odd number,” “double,” “double and halve.”

And as simple as it sounds, one student agreed with another: “I agree with [him]. $5 \times 4 = 4 \times 5$.”
Representation:

There were many examples of open array models for thinking while multiplying in this lesson. This is one of many:



For 9×14 , one student suggested a 9×4 array and “[add] 10 on the side and 9 is already on the top.”

They used 2×5 arrays to investigate multiples of 10 and support their own thinking:

S1: “What do you think our big idea is?”

S2: “We noticed 5×2 , 5×4 , 5×8”

S1: “Let’s start again.”

S2: “ 2×5 , 4×10 if we double.

S2 drew a 5×2 array.

S1: Doubling doubles the answer?

S2 drew a 5×2 array + a 5×2 array.

S2: It would be 4×5 .

S1: If we add the arrays together we’d get double the answer?

S2: Yes.

S2 drew a 5×4 array.

S2: If we add different numbers of 5×2 arrays it will always be a friendly number like 10, 20, 30, 40, 50 because each 5×2 is 10.

S1: $2 \times 5 \times 2$ would be 20, $3 \times 5 \times 2$ would be 30, etc.

S2 drew 6 5×2 arrays and then drew a 10×6 array to show $(5 \times 2) \times 6$.

S1: I agree.

The teacher used the double line model this year to explore doubles, the open number line to model subtraction strategies, and open arrays to model multiplicative thinking. And when representing their thinking on posters, she has told the children to “focus on math and writing, not art. It should be clear and concise.”

Conclusion. Throughout the lesson students demonstrated evidence of mathematical thinking. As the teacher said in the pre observation interview, participating in the math DYO initiative has impacted the development of her students' mathematical thinking. "Now when they take it, they approach it differently. They are less likely to do things in order. They are more likely to make connections. They don't do as much algorithm work. This carries over into their daily work."

Teacher 5

Lesson Summary

The lesson focused on the concept that students can build proficiency with larger multiplication facts by using the smaller facts they already know. The teacher explained that she had noticed many students using repeated addition when dealing with larger multiplication facts (8×8) and she wanted them to know they could use smaller facts to learn the bigger ones. She introduced a game she invented called “Big Meals” where a large array was the “plate” and the students needed to combine two or more “side dishes” (smaller arrays) to fill the plate. She modeled the activity with a 6×7 “plate”. She found a 5×6 “side dish” and asked the students what else she would need to cover the plate. She modeled writing the equation so it looked like this:

$$2 \times 6 = 12$$

$$5 \times 6 = 30$$

$$7 \times 6 = 42$$

The students used the 4th grade TERC Investigations array cards as the “plates” for the game, because they had the larger arrays, and then the 3rd grade TERC Investigations array cards for the “side dishes,” as those arrays were smaller. At the end of the mini-lesson explaining the procedures for the game and demonstrating how to play it, the students played with partners at the classroom tables. The teacher stayed on the rug in the meeting area for about 20 minutes after dismissing the students, and she worked with individual students who came to her voluntarily and wanted help with the activity. In the final 20 minutes of the game, the teacher moved around the room to check-in with students who were playing at the tables. She emphasized the importance of the written equation, and showed the correct use of parentheses in a horizontal format, although she allowed students to write their problems in a vertical format if they weren’t comfortable using the parentheses.

At the end of the Big Meals activity, the students returned to the rug area and the teacher conducted a follow-up activity using images of curtains with different array-patterns on an overhead projector. She showed them four different images, each one more difficult than the previous one. The more difficult images showed curtains partly opened with larger arrays. The lesson and activities lasted for one hour and fifteen minutes.

Teacher Instructional Practices that Elicit Student Mathematical Thinking

Classroom Discourse. The teacher set clear goals for the “Big Meals” activity, and explicit expectations about how to proceed with the task. She values small group work and partnerships, and created a game where students worked together on the problems. She asked students to share their mathematical ideas with her by showing how they solved a problem, by writing the problem correctly, and by proving its correctness.

Open-Ended Questions/Problems. The teacher poses questions in order to stimulate mathematical thinking such as, “What smaller plates can I use to cover my big plate? Give me a ‘thumbs up’ if you have an idea.” She also values different approaches to problem solving by asking, “Did anyone do it a different way?”

“How” and “Why” Questions. The teacher uses questions to prompt students to explain what they did and why they did it. She asks students to describe how they reasoned through a task, such as “Say I use three side dishes, how would I do this same array?” The teacher wants to know if students understand how to correctly write equations for the arrays they build, and asks several times “how do you write this?”

Mathematical Representation. The teacher clearly values the importance of correct representation of the array models (“Big Plates” and “Side Dishes”) in this activity. She demonstrates two different ways to write the problems herself, one in a vertical format, the other in a horizontal format (using parentheses).

$$\begin{array}{l} \text{Examples: } 2 \times 6 = 12 \\ \quad \quad \quad 5 \times 6 = 30 \\ \quad \quad \quad 7 \times 6 = 42 \end{array} \qquad \begin{array}{l} 7 \times 6 = (2 \times 6) + (5 \times 6) \\ \quad \quad \quad 12 \quad + \quad 30 \quad = \quad 42 \end{array}$$

The teacher also insists that students notate their equations correctly: “But remember that you need to use parentheses around each array to make the equation correct.” In addition to this, the teacher also uses another context/ visual model – the curtain arrays – to strengthen the work with arrays.

Use of Contextual Problems. The teacher is attuned to the importance of connecting and grounding conceptual understandings to everyday contexts. She developed the game “Big Meals” as a means of helping them practice using the smaller multiplication facts they know to solve bigger multiplication problems thinking this would be a relevant context. Reflecting on the activity in the post-observation interview, the teacher that there may have been some confusion from the way she used the words “plate” and “sides” interchangeably. “I also think I should have talked about bites of food instead of dollars, so a 6×7 array could be a 2×7 ‘side and a 4×7 ‘side’ which would make a 42 ‘bite’ Big Meal.” The teacher reported that she recognized her students needed to return to the array model for more practice when they were using repeated addition to determine the number of Coke bottles in an Andy Warhol print. Additionally, during the follow up activity at the end of the Big Meal game the teacher used a visual array model (the curtains) as another context to provide still more practice for her students.

Student Demonstration of Mathematical Thinking

Reasoning and Evidence. In the second part of the lesson, when the teacher showed students the images of the curtain arrays, the students had an opportunity to articulate their thinking when asked how many objects there were on the curtains. When one child said, “I think it’s 14 because I counted 2 down and I think I saw 7 across,” another child refuted this because “half of the shade is closed,” so half of the objects were obscured, thus making the total 28, not 14.

Observing, Conjecturing, and Generalizing. Little demonstration of the features in this category were noted throughout the activity other than one child making the observation and conjecture about the size of an array required to fit on her Big Meal plate during that activity. The student counted the squares remaining on the array grid and said, “I need to find a 6×4 piece, right?”

Making Connections. The Big Meal game was designed by the teacher based on the idea that students could use what they already know about smaller multiplication problems to solve bigger, harder problems. In the post-observation interview, the teacher acknowledged that her

students “already have the tools to solve these harder problems.” So, the activity was built upon the idea that they would already be making connections just by finding the smaller arrays they knew to fit into the larger arrays. Some students continued to use repeated addition, however, as was observed when one child solves 3×6 quickly but “seems to get stuck on the (4×6) portion and I [the action researcher] can see her using her fingers to skip count under her breath: ‘6, 12, 18, 24.’” Students continued to make connections to earlier understanding when confronted with new, harder problems.

Strategizing. Students were observed several times strategizing and articulating their strategies during the follow up activity with the curtain array images. When asked by the teacher how they arrived at their answers the students described their processes for each of the different array models projected on the overhead. “I saw the three pears on one side and counted four across the top, so I counted three four times.” When the teacher asked if anyone solved it another way, another child said “I counted four three times.” With the second image a student said: “There are twenty four. On one side I saw 12 diamonds and then I doubled it.”

Communication. The students were well acquainted with the routine of writing their array equations (and other math work) in their math journals. The teacher emphasized correct and precise notation of the array models by having them use equations with partial products added to partial products (with parentheses separating the sets) in a horizontal format: $(3 \times 6) + (1 \times 6) + (4 \times 3) + (2 \times 3) = 42$, or written in an “add up” vertical format: $12 \times 3 = 36$

$$12 \times 4 = 48$$

$$\underline{12 \times 4 = 48}$$

$$12 \times 11 =$$

The majority of the students observed used the partial product horizontal format in recording the arrays they worked with and, though they seemed initially inconsistent using parentheses correctly, during the course of the Big Meal activity they became more consistent at notating the problems correctly.

Representation. Similar to the Communication category, the students in this classroom were familiar and comfortable expressing their mathematical ideas and solutions as equations and it was clear to the observer that they had ample experience with mathematical notation and representation. The use of parentheses was tricky for them. During the mini-lesson / introduction to the game, the teacher modeled writing the equation using parentheses and said : “I put parentheses around the two different arrays because they tell you what you have to do first.” But then she added: “Now if you don’t want to use the parentheses because they can be a little confusing, then you can write it this way instead: [and she modeled the vertical format].” There were multiple opportunities during the game where students were observed correcting their written notation by adding parentheses.

Teacher 6

Lesson Summary

The lesson focused on having students make the connection between multiplication and division. The students had previously been studying multiplication and the teacher wanted them to see its relation to division. The lesson began with the teacher presenting two questions on a chart:

- A. Jen's box has 3 rows of chocolates. There are 4 chocolates in each row. How many chocolates are there in all?
- B. B. Jen's box has 12 chocolates total. They are organized in 3 equal rows. How many chocolates are in each row?

The teacher questioned the students about the first question and then asked them to compare the second to the first. As the students spoke she represented their thinking on the chart. The diagram on the chart and the discussion paved the way for the day's lesson: Sharing 36, or Sharing 12. Students were offered a choice of assignments. Students then went off to work in groups of 3-5. The groups of five worked with a Math coach, the groups of three worked independently. The teacher circulated amongst the groups to observe and answer questions as they worked. The groups worked for about 30 minutes before the lesson came to a close and the class moved on to reading with their kindergarten reading partners.

Teacher Instructional Practices that Elicit Student Mathematical Thinking

Classroom Discourse. The teacher engaged in this practice often. During the lesson comments such as "does anyone want to add to that" was often used to encourage students to comment on the thinking of others. This practice fit with her thinking that was stated in the interview that "I also try to get kids to think mathematically through discussions."

Open-Ended Questions/Problems. Found no examples of how we defined this term. However, the students often asked their own questions which the teacher left for them to figure out.

"How" and "Why" Questions. The teacher often engaged the class with these questions. E.g. "How does this array explain Student D's thinking?" The question that began the study was also an example of how these types of questions were used in this lesson.

Mathematical Representation. As students spoke, the teacher represented their thinking on the chart. She also encouraged them to represent their thinking to the class by saying "How can I see it? Shall I make an array?"

Use of Contextual Problems. The initial problem was one with which all could relate, sharing brownies. The subsequent problems, sharing cookies to the soccer team, were also contextual. The teacher's practice here also provided examples in other areas such as encouraging connections, acknowledging student thinking, and refocusing students. An area which was not on our list but which this teacher thinks important, is finding an entry point for all students. Her class work sheets were created so that all students were able to approach the problem in some way.

Student Demonstration of Mathematical Thinking

Reasoning and Evidence. Examples of this were apparent in the classroom. During the group lesson students made comments about what process they might use “if there are four chocolates in a row and three groups then you have to multiply” and “are they trying to find out the total or how much in each?”

Observing, Conjecturing, and Generalizing. Evidence of this was seen in students’ comments such as “you can do two rows of 3 twice, that would be doubling 6 to get 12.”

Making Connections. Was seen when students made remarks such as “in the first problem $3 \times 4 = 12$ so in the second problem the answer must be 4.”

Strategizing. Examples of this was seen in the whole group lesson as the students commented that “you can count by 4” and “if there are four chocolates in a row and three groups then you can multiply.”

Communication. Evident throughout. As the students shared ideas, others responded. The communication was verbal as in the two students who were disagreeing about whether all of the answers to the sharing 36 would be 12; or nonverbal as when students shook their heads in agreement at the meeting area.

Representation. Examples here were seen when students shared their thinking when the problem was introduced and when they worked in their small groups. As they worked students drew diagrams to explain their thinking. In addition, there was an example in which a student was not quite clear on how to proceed but tried to apply what she knew of previous concepts to the one being shown. Student G for example, knew how to add to get 12 but did not recognize the problem.

Teacher 7

Lesson Summary

The lesson focused on students visualizing dimensions of arrays as multiplication combinations, sorting through different arrays into ones they know automatically and ones they don't, and finding strategies for figuring out the ones they don't know. The class started the lesson together at the meeting area, discussing ways they see or understand arrays as groups of squares such as "6 groups of 4", and talking about how they know how many total are there, such as "if I know that 5 groups of 4 is 20, it's one more 4, so 24". Students then worked in pairs sorting through array cards to identify combinations they know and ones they are still working on, and then brainstormed ways of using what they do know to figure out the products of combinations they don't now right away. The teacher circulated during this time to support students and assess their understanding.

Teacher Instructional Practices that Elicit Student Mathematical Thinking

Classroom Discourse: There is a lot of value put on classroom discourse in this teacher's classroom. Students were given many opportunities to discuss, share and respond to each other's ideas during the minilesson time as well as in small groups. There was evidence of substantial discourse between teacher and students, as well as students with each other. There is space and opportunity in this classroom for students to disagree with or challenge each other's ideas and even to disagree with the teacher. It is clear that A great example of this is when the teacher misrepresented what one student was describing by drawing an extra column on an array, instead of a row. One student raised his hand to say, "No, you just made 28 because you only added 4 more. You need to add another column of six to make an array of 30". Students clearly feel comfortable with the notion that the teacher is not the sole arbiter of knowledge and information in the classroom. When this student corrects the teacher she doesn't seemed embarrassed or surprised and the student doesn't seem smug or arrogant. No one chuckles or seems uncomfortable with the idea that the teacher can make a mistake or that a student can correct a teacher.

Open-ended Questions: The teacher prompts student thinking by asking many open ended questions, such as "are there any other groups?", "Any other strategies?", and "how else can you figure it out?". Using prompts such as these she is leaving the door open for other possibilities and also encouraging the students who have already thought of an "answer" to continue to ponder the possibilities and push their own thinking. When working with students in small groups she scaffolds student thinking by asking open-ended questions that refer to students work they have already done. For example, when working with one student who was stuck on a combination he didn't know right away, she asked, "Is there any way you can use what you learned in the last card to help you figure out the product of this one? Can you find this array in that array?"

How and Why Questions: In one moment a student attempts a strategy for figuring out how many would be in an array of 9×5 by looking at an array of 10×5 and then taking away one group of 9. The teacher notices this and asks the student "why are you minusing a 9?". The student pauses to think about it and the teacher follows up with the questions, "where are the 9 and the 5 in this array", and hands him grid paper to model it. This scaffolding worked well for

the student to arrive at an understanding of his mistake, but had she left him to consider only the first question his realization and eventual understanding may have been deeper and more profound.

Mathematical Representation: There is a Smartboard in this classroom and the teacher depends on it heavily in the first part of the lesson to demonstrate the various models for describing arrays that students are sharing. She uses the different colored markers to highlight the different ways that students are seeing groups in an array. The students representations are all almost entirely verbal for most of the lesson, and the teacher's are verbal as well as visual. In one case when a student is having trouble coming up with a good strategy she provides him with grid paper where he outlines the groups as he sees them, and breaks arrays up into chunks that he can recognize and work with easily.

Student Demonstration of Mathematical Thinking

Reasoning and Evidence: It is clear that in this classroom there is an expectation that students will think about problems, explain their thinking and provide evidence to support their ideas. For example, one student explains why she knows 6×4 is 24 by offering the following model, "Well, I know that 6×5 is 30, so just take away one group of 6, that's 24". When asked how else one might consider this problem, another student suggests a different explanation for the same problem, also with evidence as follows: "I know $2 \times 6 = 12$, then add another 12." When the teacher mistakenly misrepresented a student's strategy, a student points out her error with evidence as to why it is incorrect, and offers a corrected strategy: "No, you just made 28 because you only added 4 more. You need to add another column of six to make an array of 30".

Making Connections: There were two great examples of students making connections in this lesson. In one example a student makes a connection to a strategy that was shared earlier by another student: She says, "He did the same thing as Serafina". She doubled the array lengthwise, He doubled it going the other way". She notices that if you "double" one dimension of an array you will get a product that is also double that of the first array. She points out that it is the "same thing" or the same product if you double the length or double the width, or likewise one of the factors in a multiplication combination. In another example a student makes a connection to a realization he had in an earlier problem. When he pulls an array card that says 4×11 on it he connects it to another, similar problem he just did and says, "Hey this is just like the other one – it's 44!"

Strategizing: Students apply and share a variety of strategies for solving multiplication problems with arrays. Some of the strategies involve doubling and/or partial products, like one student who says, "I know $2 \times 6 = 12$, then add another 12."; and another who says, "I did it by 4. I counted by 4 in three columns, that was 12, then I doubled the 12 because there is the same number on the other side – half of 24 is 12." Another student suggests that, "if you know that $2 \times 9 = 18$, then add $18 + 18$ to figure out 4×9 ." Many students used combinations they know to figure out ones they did not know right away. For example, Mohammed pulled 4×12 and stumbled – He tried to put it down and choose another, but then he was encouraged to find a way, "Well, I know 4×10 is 40, so... 56? Wait – I need two more 4s, that's 48". The same student used a similar strategy later on when he picked 9×5 and says, "55! I know 5×10 is 50 so... wait. If 5×10 is 50 I need to minus one 9. 51?" The teacher helps him to see where he made an error here, but he sticks with the strategy of starting with the known product of 5×10 and then taking away. Several

students whose thinking about multiplication is still emerging use a skip counting strategy. For example, one student pulls the card 6×3 and immediately puts his finger on the first square and starts counting squares by ones. Elaine says, “Wait – before you start counting, is there anything on this card that gives you a clue as to how many squares are in this row?” The student says, “oh, 6” (after looking at the dimensions printed on the card). He then skip counts by sixes to get 18.

Communication: During the first part of the lesson students seemed eager to communicate their ideas, evidenced by many waving hands in the air throughout the lesson. As ideas were communicated the teacher recorded students’ ideas on the Smartboard. The best example of students evaluating the thinking and strategies of others came when the teacher made an error in recording a student’s strategy and a student pointed it out by saying, “No, you just made 28 because you only added 4 more. You need to add another column of six to make an array of 30”. He not only identified the error but communicated the appropriate strategy, and launched a lively discussion. He also illuminated a very common pitfall for students when working with arrays -- confusing the rows and columns. During the partner work time, Mohammed picks 9×5 and says, 55! I know 5×10 is 50 so... wait. If 5×10 is 50 I need to minus one 9. 51? This student is on the verge of making some important connections, but is getting tripped up in his thinking. However, because he is talking through his thinking and approach it is easy to see exactly where his confusion lies and for the teacher to help get him back on track.

Teacher 8

Lesson Summary

The focus of the lesson was to encourage students to regroup the values of “stamps” from their individual face value into larger chunks or groups, and then apply sums or products to additional groups of the same size, as in doubling. [The teacher] began his lesson with a meeting about stamps to help children tap into their prior knowledge. There was *classroom discourse* as they discussed the artwork, what stamps are used for, the kinds of symbols found on them, etc. which also provided a *context* for the *problem*. He then touched on the values stamps have and how it is sometimes necessary to calculate the purchase price to buy or use them. Using an overhead projector, [the teacher] showed his 28 students seated around him in the meeting area a copy of the worksheet they were to complete. He began by showing his students the first set of stamps and said, “I am asking you to take a look at the set of stamps and tell me how you might figure out the value of the stamps.” [The teacher] asked *open ended and how and why questions* such as ‘how might you solve this?’ and called on a variety of children to provide answers and explain their methods of grouping the stamps. He used a marker to illustrate the way each child clumped the stamps and then represented the steps they followed on lines provided next to the stamps, *representing* their mental math work *mathematically*. At times he recorded exactly what the child described. At other times, he fed them language to push their thinking as in when Heaven said, “I saw four 5’s and new it was $20 + 20$,” and [the teacher] replied, “It’s like seeing 2×20 .” Students were paired and grouped by ability. They worked for about 30 minutes during which [the teacher] sat and worked with the three partners who struggle the most. There was another teacher in the room, our math coach, Shirley, and a para, circling around and asking children to explain their thinking. The lesson ended with a share in the meeting area of how students saw the groups of numbers and solved each problem. The lesson began at 10:15am and ended at 11:30.

Teacher Instructional Practices that Elicit Student Mathematical Thinking

Classroom Discourse. [The teacher] places a tremendous value on classroom discourse. In fact, it could be said that this is the core of all of the learning that takes place in this classroom. [The teacher’s] students are used to lengthy meetings in which there is a lot of speaking, listening, restating, interpreting, agreeing, disagreeing, and adding on going on. In his post observation interview, [the teacher] reiterated the importance of sharing strategies, especially efficient strategies that students use, in order to move students into new comfort zones.

Open Ended/How-Why Questions. [The teacher] is a master at posing these kinds of questions in his meetings and lessons. Not only does he ask why and how questions, he will often ask questions such as, what do you think? What did you notice? And how could you do this more efficiently? He is constantly pushing and challenging children to push themselves to think differently, to examine their own assumptions, and to question each other’s ideas. [The teacher] uses language and humor, such as when he stated in his pre-observation interview that he told a child who is resistant to math that he noticed that during recess the child used math when, feeling chilly, moving into the sun.

Mathematical Representation. [The teacher] made a huge effort and accurately recorded his students’ mathematical thinking on the overhead during the lesson for everyone to see. He

slowed children down and asked them to clearly explain how they got to certain points in their problem solving. And he shows them he cares about getting it absolutely right. When, during the first stamp problem on the overhead Rose told him that she grouped the sets of 5¢ stamps into 10's, [the teacher] outlined two 5¢ boxes horizontally. When Rose corrected him and said, "I saw them up and down," in columns, Steve erased his circles and changed them to accurately represent how Rose saw the 10's.

Use of Contextual Problems. This category should actually come first as the subject of this lesson was stamps. Although it wasn't [the teacher's] idea, (it was something he got from his attendance at the March IIM,) stamps are real and real people really use them. In addition, children know what stamps are and their purpose. [The teacher] was able to present the stamp problem in this lesson to his students in a way that made sense and had true meaning. It's a situation that they each could actually face someday and was attainable for every student in the room.

Student Demonstration of Mathematical Thinking

Reasoning and Evidence. The share portion of this lesson was key to revealing each student's demonstration of mathematical thinking and of his/her growth as a result of the lesson's activities. Having recorded the lesson and student responses, it seems the challenge for students was to solve the problems on the overhead using mental math and then to retrace their mental steps in order to articulate how they each worked through the steps to solve the problem. Visual tools were helpful, (having the stamps displayed overhead,) as was the time given to each child to fully explain their thought processes.

With the example of [the teacher's] student Ari, he stated that he felt he and his partner solved the 9's stamp problem the "best." First he supplied [the teacher] with his solution, and then explained the method he used to solve the problem. Once Ari regrouped the 9's into sets of 18, he actually used addition to get the total, adding the tens and ones in his 4 18's. However, Ari was able to clearly show his thinking to solve the problem.

Making Connections. It was Benny who recognized that the numbers in two stamp problems were related-the sets of 3's and the sets of 6's. In trying to explain the connection between the two problems, Benny got stuck and [the teacher] tried to help him articulate his thinking. [The teacher] said, "So you are saying is this $2 + 2 = 4$, this $2 + 2 = 4$, and this $2 + 2 = 4$." Then Benny continued from what [the teacher] said and said, " $12 + 4 = 16$, $16 + 4 = 20$, and $20 + 4 = 24$." Although the number of stamps in each group to make 12 changed, the product or sum of the groups were the same and made the same overall value. [The teacher's] goal was for his students to see the connections and make them, applying what they learned or calculated from one problem to the next.

Building on Prior Knowledge. [The teacher] recognized that his students knew how to skip count, and he knew that his students could add multiples of one number. What he wanted them to do in this activity is not re-add, or refigure sums of numbers but to see *blocks* of numbers and add or multiply bigger sets repeatedly. Multiple students demonstrated this ability, such as Jaylah, Heaven, Benny, Ari and even Alexa and Mali.

Strategizing. Here are some examples of different students' strategies:

Justin: I did skip counting.

Rose: I added the 5's into 10's: $5 + 5 = 10$. Another $5 + 5$ is 10. Another $5 + 5 = 10$. Another $5 + 5 = 10$. That's four tens.

Heaven: I saw 4 fives and I know it as $20 + 20$.

Sophia: $2 \times 8 = 16$, $1 \times 4 = 4$, $16 + 4 = 20$, $20 + 4 = 24$

Nico: I saw the top row as 12, so $12 + 12$ is 24.

The above are examples of children strategizing to solve the problems in the activity. A common practice in our school is to consistently honor everyone's ways of seeing and solving problems as well as everyone's answers, right or wrong. And, we always ask children to explain their strategies. Some classrooms even name strategies after children and refer to them as "Theo's way," etc. This particular lesson was especially suited to allowing students to use different methods and strategies to find the total values.

Communication/Representation. Most students at [name of school] are used to being asked to show their math work and to explain their thinking. We have always been more interested in that than the answers. Most visitors to our school are impressed by how well our students can articulate themselves and how clearly they can explain things. This lesson was another opportunity for students to practice their communication skills and to show people how they think mathematically. Showing their thinking in writing is usually more of a challenge, but the design of the worksheet used in this lesson allowed for open ended corresponding math number sentences or skip counting to take place. In addition, students got to practice both oral communication and to see how [the teacher] modeled the numerical representation in the mini-lesson. Children were able to speak in the meeting, during their work time and during the share. Lots of communication and representation took place.

Teacher 9

Lesson Summary

The lesson called for students to design boxes to fit 12 chocolates. Students had to find all the ways to design a box to fit 12 chocolates and to prove that they had found all the ways. After a class discussion to review the box design for 10 chocolates, the teacher sent students to work in partnerships to design boxes of 12. She went around the classroom visiting and questioning partnerships as they were working. At the end of the period, she gathered students to discuss that some partnerships were finished with work and other partnerships needed more time to finish their work. The talk turned to students articulating what their next steps had to be.

Teacher Instructional Practices that Elicit Student Mathematical Thinking

Classroom Discourse. There are a lot of opportunities for students to talk in the classroom. At the beginning of the lesson the teacher says, “We talked about Mr. Joaquin, what does he do?” This line of questioning speaks about the teacher’s belief that, “you must have some sort of understanding of the concept in order to reason about it.” In her introduction to the task, the teacher makes sure that the students understand what they need to do. She makes sure that students use precise language to describe mathematical thinking. The teacher said, “We have to use math language. A student speaks back: “The columns go up and down and the rows go side to side. The rows and columns have to have the same amount.” The teacher has students clarify thinking for each other. The teacher points to a 2×5 box and asks, “What do we know about this?” A student responds, “In every row there are 5 chocolates and 5 and 5 is 10. The teacher asks, “What is J trying to say?” A student replies, “If there are 2 rows and there are 5 in each row, he is doubling and it makes 10”. Classroom talk also happens in partnerships. Partnerships are encouraged to talk and negotiate meaning together. In the introduction of the context, the teacher said that partnerships needed to discuss what would go on poster and also provided an extension to prove they had all the ways of making the boxes. Teacher believes the questioning they were doing themselves and with their partners is an example of math thinking. The teacher holds congresses or group shares. She feels “a congress is a helpful way to promote math thinking. It is a process is to collectively learn from presenters about different approaches and strategies.” “I like the congress that we do. It takes a lot of time but it’s a big step for kids to realize that they have to listen and put in their own words what others say. I am beginning to see the results of every congress we’ve had. I can see that they’re able to listen and question and are on their way to good mathematical thinking. They are able to offer the presenters suggestions, questions and comments.. this shows that they’re exhibiting mathematical thinking.”

Open-Ended Questions. The problem for exploration was: “Joaquin needs to organize his chocolates and he needs us to help him think about how to organize 12 chocolates. You are going to work on showing how to show and develop all of the boxes for 12. Then you are going to think about how you know that you have found all the ways.” The teacher said, “we began doing these kinds of open units of study. They are open and allow kids to explore problems with others, to problem-solve with others. The open investigations help promote math thinking. The congress or share is a helpful way to promote math thinking, process allows students to collectively learn from presenters. Students discuss different approaches and strategies. Because the problems are open, in the sense that you present problems in context and they go off and try

to find a way to solve it, they're forced to use what they know and build from there.”
“Mathematicians learn in groups instead of individually by understanding each others’ work.”

“How” and “Why” Questions. In her interview the teacher said, “Mathematical thinking is a process, kids won’t develop this if teachers don’t get it. Teachers won’t know the questions to ask to get certain responses from kids if they don’t understand the concepts. Teachers need to ask questions that help them realize and use what they know, questions that help them build upon what they know, use what they know, looking at others’ work. Questions should touch upon the various levels of math thinking, the steps in the process of mathematical thinking”. .” When conferring with partnerships, the teacher changed her questioning to clarify and extend the thinking that partnerships were doing. She refocuses them. There is a 2x5 box and she asks a student how many are in the box. He says 10 and she asks if it would fit 12. The student says no so the teacher asks, “What do you need to add to the box to make it fit 12? He says, “Add a square” and she clarifies and asks, one little square? and he says,” No row.” The teacher sets up partnerships. Students in partnerships are expected to ask questions of each other.” If you are working in partnership you can ask: Do you understand what he/she did? Can you say in your own words? What can you say about his/her approach?”

Mathematical Representation. The teacher introduces the next step in the problem they are solving. In her introduction she refers to previous work done around this idea of boxes. She begins with the basic model of a box design, a 2x5 array. “We started helping Mr. Joaquin with the number 10. (the teacher points to a 2x5 box and asks “what do we know about this?”) Students will use this array model to represent their thinking about designing chocolate boxes to fit 12 chocolates. One group draws their boxes on a chart paper and shows that there is a connection between 2 boxes and represents this by drawing an arrow between them. This group is also labels their boxes by naming the number of rows and columns in each box.

Use of Contextual Problems. The teacher expressed enthusiasm in using contextual problems. As stated previously the teacher said, “Since we began doing certain units of study, those units are open and allow kids to explore problems with others, to problem-solve with others. The open investigations help promote math thinking”. The teacher said, “As a process the first step is to have that understanding of the concept you’re exploring, what information the problem is giving you, you need to understand it, you need to make connections between what I know to help me figure out the relationship between what I know and what I am exploring. Once you problem solve and decide on a strategy, you need to take that and apply it in the real world. Math thinking is evident if child can take the thinking out and use it in the real world. That’s the final stage-when they apply it in the real world”. The teacher also adds that “because the problems are open , students go off and try to find a way to solve it and they’re forced to use what they know and build from there, onto that.”

Student Demonstration of Mathematical Thinking

Reasoning and Evidence. Students were looking for ways to make boxes to fit 12 chocolates. Using the array model, students charted their thinking. In one instance, a student is thinking through the idea of the commutative property and concludes, says “it’s the same if you turn it”. Another partnership has 2x6 and 6x2 next to each other and they say it’s because they’re connected and that’s why they have drawn them next to each other. They also draw an arrow between them to show the connection. Students are using past knowledge to think about how to

articulate the total number of chocolates. Student says: “In every row there are 5 chocolates and 5 and 5 is 10.” Teacher asks, “What is J trying to say?” Student says, “If there are 2 rows and there are 5 in each row, he is doubling and it makes 10.”

Making Connections. Students are beginning to grapple with the idea of the commutative property. The teacher said, “We would call this a 6x2 box (she says as she points to the lines and columns) and she points to the 2x6 box. Then she asks the student to name it, drawing their attention to the fact that it is turned. As previously stated, Students drew boxes next to each other because they said the boxes were connected. They also illustrated connection with the use of an arrow. Teacher said in an interview: “Mathematical thinking is more of a process where anyone shows their ability to reason, to problem solve, to make connections between the concepts and ideas they know and build relationships between those concepts.”

Communication. Teacher sets up the expectation for work to be shared in the community. Teacher says: “Remember that when we are showing our work, we need to be clear. Here I see a connection clearly, but here, (pointing to 2 very spaced out boxes) what can we do to show they are connected?” “These boxes that are so far apart can be confusing to someone reading your paper.” Students say, “We have learned that when we find a connection, we draw an arrow between the 2 boxes”. Student says: “we found four ways, but when we turned it, it was a connection...” Teacher said, “I think that they had good conversations during work that demonstrated math thinking, their conversations were full of good math words like columns, rows, connections, rotate. They had to make decisions about what to put on their papers and what to call it. In the share when I asked them what their next steps would be, they had an idea about what they would do.” “Having conversations and negotiating through different ideas demonstrated mathematical thinking. The questioning they were doing themselves and with their partners is an example of math thinking.”

Teacher 10

Lesson Summary

The lesson focused on students identifying situations as multiplication or division situations and then using strategies to solve them. To start the lesson, the teacher posed a problem to students and asked them to determine whether it was multiplication or division. Then students came to the board to show their strategies to the class. In small groups, students solved multiplication and division situations with peers at their table groups. In the wrap-up meeting, students played a game called Missing Factors and made connections between the problems they had solved and the game.

Teacher Instructional Practices that Elicit Student Mathematical Thinking

Classroom Discourse. The teacher values classroom discourse in teaching mathematics. In the lesson, classroom discourse occurred in the meeting area as students shared strategies for solving the problem that was presented at the start of the lesson. The teacher asked students if they agreed or disagreed with each other in order to promote student discourse. She always took a neutral stance, encouraging students to discuss and argue about mathematical ideas. Students also shared their ideas in a turn and talk, which they do regularly during math lessons in the meeting area. During small group work, students freely shared questions and ideas. For instance, one student helped another student create a number sentence to accompany a division situation. Also during small group work, the teacher encouraged discourse by asking students to share their strategies with their table-mates.

When speaking about classroom discourse, the teacher said, “One [strategy I use for promoting discourse] is presenting a problem and asking students their strategies and asking them to talk with one another; think about whether they agree. Never telling them, asking other students if they agree, and see what they notice. Never say no, but say, ‘Hmp! What do other people think? What are your thoughts?’ Really encouraging that kind of discussion about the problem.”

Open-Ended Questions. The teacher asks open ended questions to stimulate mathematical thinking. For instance, the teacher asked students whether they agreed if the situation was a division situation and why. When circling around the room during small group work, the teacher asked questions such as, “Why did you do that?” in order to encourage students to explore mathematical ideas and make sense of their strategies.

“How” and “Why” Questions. The teacher asks how and why questions to give students an opportunity to describe what they did and why they did it. For instance, when playing Missing Factors, the teacher asked how a student can figure out the missing factor. In this situation, she required the student to support her answer with evidence. Students in the class are used to giving evidence to support their thinking and therefore understand that mathematics is not just about the “right answer.” The teacher also regularly asks students to describe why a student is using a particular strategy to involve the entire class in reasoning and mathematical thinking. This method also furthers dialogue among the students in her class.

Mathematical Representation. The teacher encourages multiple student representations and facilitates discussion about these representations so that students can express their mathematical ideas. For instance, for the problem presented at the beginning of the class period, the teacher encouraged a student to share her representation—drawing hash marks inside three circles to divide 18 by 3. The teacher then encouraged another student to represent his thinking when he wrote a number sentence on the board. The teacher modeled her own mathematical thinking during the Missing Factors game when she pointed to a dimension on an open array she had drawn on the board and asked out-loud, “What is this telling me?” The teacher regularly encourages the use of visuals and manipulatives so that students can think about a problem or concept. For instance, the open array encouraged students to make connections between arrays, skip-counting and multiplication. When the teacher wrote on the board $5 \times \underline{\quad} = 35$, she encouraged students to think about the relationship between factors and dimensions. Students then skip counted by fives to 35 in order to find the missing factor, 7. Another student made the open array a closed array by drawing lines within the array and realized there were 7 columns. Different representations used in this situation encouraged students to find the missing factor using a variety of methods.

Use of Contextual Problems. The teacher believes in teaching mathematical thinking as it is necessary beyond the classroom. Therefore, many story problems used in lessons and classwork involve real life situations that present mathematical problems in the world. The teacher used a situation about sharing 18 markers equally among 3 people. These types of problems are important as it helps students make connections between their developing math skills and real life contexts.

Student Demonstration of Mathematical Thinking

Reasoning and Evidence. In the lesson, students were required to demonstrate reasoning and evidence to support their ideas. For instance, when a student was asked to prove why she thought a problem was division and not multiplication, she said, “Because you have to split 18 in 3 ways.” Students are often asked to justify their answers to the class. For the Missing Factors game, a student proved her answer, 7, by making lines in the open array and skip counting by 5 to get to 35. Another student had a different way of proving his answer. He wrote 5, 10, 15, 20, 25, 30, 35 go show that $5 \times 7 = 35$. When the teacher asked what strategy he was using, a third student provided reasoning for the answer. Students are often asked to explain not just their own strategies but the strategies of their classmates, thus deepening their reasoning skills even more.

Observing, Conjecturing and Generalizing. Students often look for patterns within data and attempt to conjecture a possible general idea. For instance, when solving the last problem on her worksheet, a student noticed that she shared 24 flowers by dividing 24 hash marks in 4 circles and therefore conjectured that she needed to write a division number sentence (as opposed to a multiplication number sentence). She was able to observe the patterns of the story problems earlier in the worksheet and generalize from those cases to formulate this new concept.

Making Connections. The teacher encourages her students to use their past experiences in order to develop new understandings. For instance, a student used a fact that she had learned at her table group to help her with the Missing Factors game later in the meeting area. She said, “One of the problems [on the worksheet] was 5×6 and that was 30 so $5 \times 7 = 35$.” She used a fact that she knew to help her reason through a new situation. Earlier in the lesson, students skip counted

and made the connection between multiplication and division. They skip counted by 3's to get to 18 and used that knowledge to formulate their understanding that $18 \div 3 = 6$.

Strategizing. The teacher places a high level of importance on students working towards a logical or efficient strategy for problem solving. In the lesson, two students came to the board to solve a problem using two different strategies—the first student used a model and the second student created a number sentence. Students also used various strategies to find the missing factor, such as using a model and skip counting. During small group work, some students drew circles with hash marks to divide. Another student at the same table group skip counted instead, thereby using a more efficient strategy. In this classroom, understanding more than one strategy is valued, as opposed to just the answer, and beyond that, working towards the most efficient, or logical, strategy is one of the teacher's goals for her students. When discussing the effects of the DYO, the teacher said, "The students are certainly learning from each other and seeing there are different ways of doing things. And seeing the variety of strategies that there are to solve one problem and to come to the same solution and then beyond that to see that certain strategies may be more efficient or logical than others."

Communication. It is very important to the teacher that students communicate about mathematical ideas to peers and teachers. For instance, in the lesson, they communicated informally in pairs (called a "turn and talk") to decide upon the type of problem that was on the board. Turn and talks occur often in the classroom. Students also informally talk at their table groups during classwork. In the lesson observed, students at their table groups communicated about whether the problems were multiplication or division. One student summarized the problem by saying, "There are 2 kids and 30 pretzels. How many pretzels should each student get? What plus what equals 30?" Her table-mate replied, " $15 + 15 = 30$." The first student said, "Yes, is that multiplication or division?" Together, these two students talked through the problem and the student who was unsure at first grew to understand that the situation was a division situation. During the small group work, students not only informally talked about the problems but also expressed their mathematical ideas through number sentences and wrote them on their papers. In this class, students often show their mathematical thinking on the board, and communicate to the class about their strategies. Students also take regular assessments, where they are required to show and often explain their strategy in a written format.

Representation. Students use representations to organize, record and communicate mathematical ideas. For instance, in the lesson, students used diagrams to solve division problems (circles and hash marks) as well as wrote number sentences to represent their mathematical ideas. For the Missing Factors game, a student skip counted from 5 to 35 and represented his mathematical thinking by making an organized list of these multiples on the board. Students regularly use manipulatives in the classroom as well (such as pattern blocks and base-10 blocks), though did not use them for the lesson that was observed.